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Cycle type and Kazhdan Lusztig R-polynomials

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Contents

Introduction

In this thesis we exhibit some explicit formulas for Kazhdan-Lusztig R-polynomials for the symmetric groups.

The theory of the Kazhdan-Lusztig R-polynomials arises from the Hecke algebra associated to a Coxeter group W (see e.g.[10], Chap.7) and was introduced by Kazhdan and Lusztig ([11], Sect.2) with the aim of proving the existence of another family of polynomials, the so-called Kazhdan-Lusztig polynomials. The R-polynomials, as the Kazhdan-Lusztig polynomials, are indexed by pairs of elements of W and they are related to the Bruhat order of W. Most of the importance of these polynomials comes from their applications in different contexts, such as representation theory, topology and the algebraic geometry of Schubert varieties (see e.g. [12], [20], [5] and [1]). Moreover the importance of the R-polynomials stems mainly from the fact that they allow the computation of the Kazhdan-Lusztig polynomials.
Although the explicit calculation of the R-polynomials is easier than that of the Kazhdan-Lusztig polynomials, one encounters hard problems to find closed formula for them, even when W is the symmetric group. In recent years purely combinatorial rules to compute the R-polynomials have been found, (see, e.g. [5], [4]). A natural approach to this matter consists in the investigation of the R-polynomials associated to pairs of permutations (u, v), where v is obtained from u by applying particular permutations, since every pair can be viewed in this way.

Such an approach is in the paper [3] in which the author gives an explicit formula for the pair (u, wu), where w is an increasing subsequence of the permutation u. From this theorem follows easily that the R-polynomial of a pair of permutations (u, v), where $v = u(i, j)$, has a nice factorization. Moreover from the same result we derive closed formulas for the pairs of permutations (u, v), where $v = u(a, b, c)$, as we will see in Chapter 2 of this thesis.

One of the contributions of this work is the introduction, in Chapter 2, of an equivalence relation between pairs of permutations which preserves the R-polynomial. This relation indicated with $\sim$ can be defined in the more general context of the finite Coxeter groups and it implies the conjugacy relation in every Weyl group: let $u, v, x, y \in W$ be such that $(u, v) \sim (x, y)$ then $[u^{-1}v]_c = [x^{-1}y]_c$, (here $[\ ]_c$ is the

conjugacy class). The proof of this result is based on a deep result of Carter [7]. This relation allow us to reformulate the problem of finding explicit formulas for the R-polynomials in the symmetric group in terms of cycle-type or, equivalently, of partitions and more generally, for each Weyl group, in terms of the R-polynomials of an equivalence class instead of a pair of elements. If we focus our attention on a particular cycle type, we discover that the R-polynomials split this class via the equivalence classes related to this cycle type. It is clear that the research problem that we are studying is equivalent to the comprehension of how the conjugacy classes are split. We first had the idea that the answer was on the occurrence of certain patterns in the permutation u of a pair (u, v), taken in an equivalence class on $\sim$. Unfortunately, the investigation on pattern of length 3 gives a negative answer. Nevertheless, in this case we proved that with some additional condition on u, it is possible to obtain other classes with explicit R-polynomial factorization, see Proposition 3.14.
We also introduce a classification of the equivalence classes which depends on a result contained in [3], which is Theorem 1.21 of this thesis: we divide these classes in irreducible and reducible.
The irreducible classes are the first object of our investigation: if we obtain explicit formulas for these then we can obtain explicit formulas for infinitely many reducible classes. In Section 2.2 we present two irreducible classes, one related to the partition $(2, 1, 1, \dots)$ and one to the partition $(3, 1, 1, \dots)$, and we prove, as said before that the R-polynomials related to these classes have a closed product formulas. In the same section we obtain from these other explicit formulas for reducible classes related to the partitions $(2, 2, 1, \dots)$, and $(3, 2, 1, 1 \dots)$, but we also give the idea that there are other reducible classes that we can cover by a combination of the results for these one irreducible, as in Proposition 3.14 or 3.21; this means, more generally, that the knowledge of certain explicit formulas allows the computation even of the R-polynomials of not equivalent pairs.
In Chapter 3 we prove our main results: the explicit formulas for the family of permutations which contains a pattern $k12 \dots k-1$ and pattern $23 \dots k1$. The proof carries over by showing that $(u, v) \sim (x, (1, n))$, where x is a k-cycle and the $R_{x,(1,n)}(q)$ can be directly computed. We treat explicitely the cases for k=2,3,4, giving closed product formulas also for classes which are equivalent to the pairs (u, v) in that u contains a pattern $k12 \dots k-1$ or $23 \dots k1$.
Moreover, for case k=3,4 we apply these results to obtain explicit formulas for pairs which are not equivalent to the one descrived in the main results but their R-polynomial is a linear combinations on $\mathbb{Z}[t]$ of R-polynomials of classified classes.
In the last section of Chapter 3 we explain that we have found more closed product formulas than the ones we have explicitly written, since the knowledge of certain explicit formulas can produce other formulas in three ways which are developed in this thesis: by equivalence, by reducible classes , by linear combination.

The organization of this thesis is the following.
In the first chapter we recall some basic definitions, notation, and results. We define the Hecke algebra of a Coxeter groups and then the Kazhdan-Lusztig R-polynomials. We collect their fundamental properties and some results on their calculation for certain classes of permutations. In the second chapter we introduce an equivalence relation on $W \times W$ which preserves the R-polynomials and we give explicit formula for the conjugacy classes of the transpositions and three-cycles. In Chapter 3 we exhibit the closed product formulas for the R-polynomials for the classes of permutations described above. In the last chapter we also give some open problems and conjectures.

Chapter 1

Kazhdan-Lusztig R-polynomials

In this chapter we define the family of Kazhdan-Luszitg R-polynomials in the general context of the Coxeter Groups. We will emphasize useful combinatorial characterization of some general properties of the symmetric group.

1.1 Notation and preliminaries

We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, 3, \ldots, a\}$, where $[0] \stackrel{\text{def}}{=} \emptyset$. Given n, $m \in \mathbf{P}$, $n \leq m$, we let $[n, m] = [m] \setminus [n-1]$. We write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted with $|A|$. Given a set X we will let $S(X)$ be the set of all bijections of X in itself and $S_n \stackrel{\text{def}}{=} S([n])$.

If $\sigma \in S(X)$ and $X = \{x_1, \ldots, x_n\}_< \subseteq \mathbf{P}$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(x_i) = \sigma_i$, for $i = 1, \ldots, n$, and we call this *complete notation.* If $\sigma \in S_n$ then we will also write σ on *disjoint cycle form*, (see, e.g.[21], Sect 1.3) and we will not usually write the 1-cycles of σ. For example, if $\sigma = 365492187$ then $\sigma = (1, 3, 5, 9, 7)(2, 6)$. Given $\sigma, \tau \in S_n$ then $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(1, 4) = (1, 4, 2)$.

We recall here the definition of a Coxeter group, but we refer to [10] for general Coxeter groups notation and terminology

Let S be a finite set and $m : S \times S \to N \cup \{\infty\}$ the Coxeter matrix defined by:

- $m(s, s') = 1$ if and only if $s = s'$
- $m(s, s') = m(s', s)$, $\forall s \neq s'$
- $m(s, s') \geq 2$ $\forall s \neq s'$

The group with the presentation:

$$\begin{cases} \text{Generators} : S \\ \text{Relations} : (ss')^{m(s,s')} = e, \forall s, s' \in S,\ m(s,s') < \infty \end{cases}$$

is the **Coxeter group** W **generated by** m **and** S.

We denote by e the identity of W. The pair (W, S) is said to be a Coxeter system. We give the definition of a class of Coxeter Group which will be used in Chapter 2.

Definition 1.1. *A Coxeter group is a* **Weyl group** *if*

1. *W is finite*
2. *$m(s, s') \in \{2, 3, 4, 6\}, \forall s, s' \in S, s \neq s'$*

A very important notion is the **length function**, that is the following.

Definition 1.2. *Let (W, S) be a Coxeter system. Each element $w \in W$ can be written as a product of generators:*

$$w = s_1 s_2 \dots s_k, \quad s_i \in S.$$

If k is minimal among all such expressions for w then k is called **length of** *w (written $\ell(w) = k$) and the word $s_1 s_2 \dots s_k$ is called a reduced word (or reduced decomposition or reduced expression) for w.*

We let "$s_1 s_2 \dots s_k$" denote both the product of these generators (an element of W) and the word formed by listing them in this order. By convention, $\ell(e) = 0$, and e can be viewed as the "the empty word".
Here we have some basic properties of the length function.

Proposition 1.3. *For all $u, w \in W$:*

(i) $\ell(uw) \equiv \ell(u) + \ell(w) \pmod 2$;

(ii) $\ell(w^{-1}) = \ell(w)$;

(iii) $\mid \ell(u) - \ell(w) \mid \leq \ell(uw) \leq \ell(u) + \ell(w)$;

(iv) $\ell(uw^{-1})$ *is a metric on* W;

(v) $\ell(ws) = \ell(w) \pm 1,\ s \in S$.

See [10], Chapter 7, for a proof.
Now we can give the definitions of **right descent** and of **left descent** of an element $\sigma \in W$; these are very important for the definition of the Kazhdan-Lusztig R-polynomials.

Definition 1.4. *Given a Coxeter system (W, S) and $\sigma \in W$ we define*

$$D(\sigma) \stackrel{\text{def}}{=} \{s \in S : \ell(\sigma s) < \ell(\sigma)\}, \textit{ the set of right descents of } \sigma;$$

$$D_L(\sigma) \stackrel{\text{def}}{=} \{s \in S : \ell(s\sigma) < \ell(\sigma)\}, \textit{ the set of left descents of } \sigma.$$

An element of $D(\sigma)$, i.e a right descent, is usually called *descent.*
We let $T \stackrel{\text{def}}{=} \{wsw^{-1} : s \in S, w \in W\}$, which is called the *reflection set of W.*

We will always assume that W is partially ordered by (strong) **Bruhat order** which is defined as follows:

Definition 1.5. *$u, v \in W$, $u \leq v$ iff $\exists t_1, \ldots, t_r \in T$, for $r \in \mathbf{N}$ such that:*

(i) $v = ut_1t_2 \ldots t_r$

(ii) $\ell(ut_1 \ldots t_{i+1}) > \ell(ut_1 \ldots t_i)$ *for* $i = 0, \ldots, r-1$.

We refer the reader to [10], Sect 5.9, for the properties of Bruhat order, here we recall the characterization of this ordering in terms of **subexpressions** of a given reduced expression $w = s_1 \ldots s_k$, by which we intend subwords of the form $s_{i_1} \ldots s_{i_h}$, $(1 \leq i_1 < i_2 < \cdots < i_h \leq k)$.

Theorem 1.6. *Let $w = s_1 \ldots s_k$ be a fixed, but arbitrary, reduced expression for w. Then $w' \leq w \iff w'$ can be obtained as subexpression of this reduced expression.*

We will use this characterization in the following chapters, particularly in Section 2.3.

1.2 Combinatorial description of the symmetric groups

Let $W = S_n$ then $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i \stackrel{\text{def}}{=} (i, i+1)$, for $i \in [n-1]$. For this Coxeter group combinatorial descriptions of length function, descent set and Bruhat order are well known and these are the objects of this section.

Proposition 1.7. *Let $w \in S_n$, and $i \in [n-1]$. Then*

(i) $\ell(w) = inv(w) \stackrel{\text{def}}{=} |\{(i,j) \in [n] \times [n] : i, j, w(i) > w(j)\}|$*; the number inv(w) is usually known as inversions of w.*

(ii) $s_i \in D(u) \iff u(i) > u(i+1)$.

(iii) $s_i \in D_L(u) \iff s_i \in D(u^{-1})$.

We refer the reader to [15] for a proof.

For example, if $u = 13524$ then $inv(u) = |\{(2,4),(3,4),(3,5)\}| = 3$, $D(u) = \{(3,4)\}$ and $D_L(u) = \{(2,3),(4,5)\}$.
In the rest of this thesis a descent $(i, i+1)$ may be written briefly as i.

We also introduce a distance on S_n, since we will prove one result of Chapter 3, by induction on it. This function plays an important role in [3].

Definition 1.8. *Let $u, v \in S_n$, then*

$$d(u,v) \stackrel{\text{def}}{=} max\{i \in [n] : u^{-1}(i) \neq v^{-1}(i)\}$$

where, by definition, $max\{\emptyset\} = 0$.

For example, if $u = 13457286$ and $v = 12367548$, then $d(u,v) = 8$.

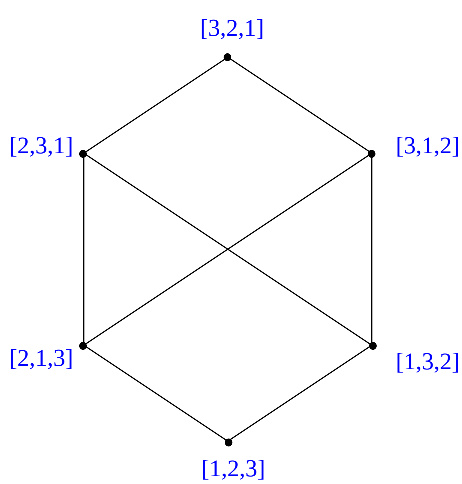
We give now the characterization of the Bruhat order.
For $u \in S_n$ and $i \in [n]$, let $\{u^{i,1}, \dots, u^{i,i}\}_< \stackrel{\text{def}}{=} \{u(1), \dots, u(i)\}$.

Theorem 1.9. *Let u, $v \in S_n$. Then $u \leq v$ iff $u^{i,j} \leq v^{i,j}$ for every $1 \leq j \leq i \leq n-1$.*

A proof of this result can be found in [15].
The following is an example of application of Theorem 1.9. Let $u = 14325$ and $v = 52341$ then

$$(u^{1,1}, u^{2,1}, u^{2,2}, u^{3,1}, u^{3,2}, u^{3,3}, u^{4,1}, u^{4,2}, u^{4,3}, u^{4,4}) = (1,1,4,1,3,4,1,2,3,4)$$
$$(v^{1,1}, v^{2,1}, v^{2,2}, v^{3,1}, v^{3,2}, v^{3,3}, v^{4,1}, v^{4,2}, v^{4,3}, v^{4,4}) = (5,2,5,2,3,5,2,3,4,5)$$

so $u < v$.
Here is the Bruhat order of S_3:



We recall that there exists a maximal element under the Bruhat order, (see [10] chap. 5) this is the permutation $w_0 \stackrel{\text{def}}{=} n(n-1)(n-2)\dots 321$.
The element w_0 acts, by multiplication, on the elements of S_n as follows: $\forall u \in S_n$, we have that $w_0 u = n+1-u(1)\dots n+1-u(n)$ and $uw_0 = u(n)\dots u(1)$. Moreover the multiplication by w_0 is an anti-automorphism of Bruhat order:

Proposition 1.10. *Let $u, v \in S_n$. Then the following conditions are equivalent:*

(a) $u < v$

(b) $u^{-1} < v^{-1}$

(c) $vw_0 < uw_0$

(d) $w_0 v < w_0 u$

This follows from Theorem 1.9, (see e.g. [15]).

1.3 Kazhdan-Lusztig R-polynomials

We now introduce the family of *R-polynomials of* W and to do that we first define $\mathcal{H}(W)$ the *Hecke algebra* associated to (W, S), (see, e.g.[10], Chap.7) as the free $\mathbf{Z}[q, q^{-1}]$-module having the set $\{T_w : w \in W\}$ as a basis multiplication such that

$$T_w T_s = T_{ws}, \quad if \quad \ell(ws) > \ell(w)$$
$$T_{s^2} = (q-1)T_s + qT_e$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as unity and that each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see, [10], Proposition 7.4).

Proposition 1.11. *Let $v \in W$. Then*

$$(T_{v^{-1}})^{-1} = (-1)^{\ell(v)} q^{-\ell(v)} \sum_{u \leq v} (-1)^{\ell(u)} R_{u,v}(q) T_u$$

where $R_{u,v}(q) \in \mathbb{Z}[q]$.

The polynomials defined by the previous proposition are called Kazhdan-Lusztig R-polynomials of W. It is easy to see that $R_{u,u}(q) = 1$, and it is customary to let $R_{x,w}(q) = 0$, if $x \not\leq w$.

We give now a fundamental property of this polynomials which often is used to define them

Theorem 1.12. *Let $u, v \in W$ such that $u \leq v$. Then, for every $s \in D(v)$, we have that*

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q), & \text{if } s \in D(u) \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q), & \text{if } s \notin D(u). \end{cases}$$

See [10], Sect. 7.5 for a proof. This theorem gives an inductive procedure to compute the R-polynomials of W since $\ell(ws) < \ell(w)$.

Moreover it can be used to prove by induction on $\ell(v)$ the following basic properties:

Proposition 1.13. *Let $u, v \in W$, $u \leq$. Then $R_{u,v}(q)$:*

1. *is monic with degree $\ell(v) - \ell(u)$;*
2. *has $(-1)^{\ell(v)-\ell(u)}$ as constant term.*

We note that a left version of Theorem 1.12 holds.

Theorem 1.14. *Let $u, v \in W$ such that $u \leq v$. Then, for every $s \in D_L(v)$, we have that*

$$R_{u,v}(q) = \begin{cases} R_{su,sv}(q), & \text{if } s \in D_L(u) \\ qR_{su,sv}(q) + (q-1)R_{u,sv}(q), & \text{if } s \notin D_L(u). \end{cases}$$

We see the following example of computation of an R-polynomial for $W = S_n$. We calculate $R_{14325,52341}(q)$ by using Theorem 1.12. First we note that from Theorem 1.9 we know that $14325 < 52341$, so this polynomial is not zero:

$$R_{14325,\mathbf{52}341}(q) = qR_{41325,25341}(q) + (q-1)R_{14325,2\mathbf{53}41}(q) = 0 + (q-1)R_{13425,23\mathbf{54}1}(q) =$$
$$= (q-1)R_{13245,234\mathbf{51}}(q) = (q-1)(qR_{13254,23415}(q) + (q-1)R_{13245,23\mathbf{41}5}(q)) =$$
$$0 + (q-1)^2(qR_{13425,23145}(q) + (q-1)R_{13245,2\mathbf{31}45}(q)) =$$
$$(q-1)^3 R_{12345,\mathbf{21}345}(q) = (q-1)^3(qR_{21345,12345}(q) + (q-1)R_{12345,12345}(q) = (q-1)^4$$

At each step we have written in bold letter the descent of the greatest permutation, respect to which we apply Theorem 1.12.

A fundamental property of the R-polynomials associated to a finite Coxeter group is given in the next

Proposition 1.15. *Let $x, w \in W$, where W is a finite Coxeter group and w_0 its longest element then*

$$R_{x,w}(q) = R_{x^{-1},w^{-1}}(q) = R_{w_0 w, w_0 x}(q) = R_{w w_0, x w_0}(q).$$

(See [10], Proposition 7.6).

We introduce the family of $\tilde{R}_{u,v}(t)$, which gives a combinatorial interpretation of the R-polynomial of W,

Theorem 1.16. *Let $u, v \in W$; then there exists a unique polynomial $\tilde{R}_{u,v}(q) \subseteq \mathbf{N}[q]$ such that*

$$R_{u,v}(q) = q^{(\ell(v)-\ell(u))/2}\tilde{R}_{u,v}(q^{1/2} - q^{-1/2}).$$

From Theorem 1.16 and Theorem 1.12 follows that

Theorem 1.17. *Let $u, v \in W$ such that $u \leq v$.Then, for every $s \in D(v)$, we have that*

$$\tilde{R}_{u,v}(t) = \begin{cases} \tilde{R}_{us,vs}(t), & \textit{if } s \in D(u) \\ \tilde{R}_{us,vs}(t) + t\tilde{R}_{u,vs}(t), & \textit{if } s \notin D(u). \end{cases}$$

Theorem 1.17 gives an inductive procedure to compute $\tilde{R}_{u,v}(t)$ since $\ell(v(i, i + 1)) = \ell(v) - 1$, assuming by definition that $\tilde{R}_{u,v}(t) = 0$ if $u \neq v$ and $\tilde{R}_{u,v}(t) = 1$ if $u = v$.

We note that Theorem 1.16 permits to work with the $\tilde{R}_{u,v}(t)$ since every result can be easily translated in terms of R-polynomials, with the advantage that they have positive coefficients instead of integer coefficients as the R-polynomials have.

The $\tilde{R}_{u,v}(t)$ satisfy the same important rule given in 1.15, we give the version for the symmetric group which is the one that we will use to prove our main results.

Proposition 1.18. *Let $u, v \in S_n$; then*

$$\tilde{R}_{u,v}(t) = \tilde{R}_{u^{-1},v^{-1}}(t) = \tilde{R}_{w_0v,w_0u}(t) = \tilde{R}_{vw_0,uw_0}(t).$$

The above result can be proved using properties of Hecke algebra and 1.16, (see [10], Proposition 7.6).

The left version of Theorem 1.17 follows easily from 1.18:
Let $u, v \in S_n$ such that $u \leq v$. Then, for every $s \in D_L(v)$, we have

$$\tilde{R}_{u,v}(t) = \begin{cases} \tilde{R}_{su,sv}(t), & \text{if } s \in D_L(u) \\ \tilde{R}_{su,sv}(t) + t\tilde{R}_{u,sv}(t), & \text{if } s \notin D_L(u). \end{cases}$$

We end this section mentioning the direct connection between the R-polynomial of the symmetric group and algebraic geometry.
We refer to [15], Appendix; [1]; or [9] for the notion of Schubert cell. For $w \in S_n$ let Ω_w be the *Schubert cell* indexed by w, then the following result holds:

Theorem 1.19. *Let F be a finite field of order q, then*

$$R_{u,v}(q) = |\Omega_u \cap \Omega_v^*|$$

where Ω_v^* is the opposite Schubert cell to Ω_v

This is a consequence of the main theorem of [8], see also [20].
A curiosity note is that the historical reason for which we use the variable "q" for the R-polynomials is the geometrical meaning given in the above theorem.

1.4 Some results on the explicit calculation of the R-polynomials

In this section we survey some results on the explicit calculation of the R-polynomials, before this we note that a general closed formula for the R-polynomials does not exist; for example,

$$\tilde{R}_{12345,54321}(t) = t^2(1 + 5t^2 + 10t^4 + 6t^6 + t^8)$$

and

$$\tilde{R}_{123456,654321}(t) = t^3(1 + 9t^2 + 39t^4 + 57t^6 + 36t^8 + 10t^{10} + t^{12}),$$

and these factors are irreducible over the field of rational numbers.

Although, the explicit calculation of the R-polynomials is not possible in general there are several general classes of permutations for which explicit formulas exist,(see, e.g. [3],[20] and [17]); some of them are related to the pairs of permutations (u, v) in which v is obtained from u by applying a particular permutation, these are the main reaserch object of this thesis.
We give here the Brenti's theorems which consider these kind of pairs of permutations and on which are based some results of Chapter 2.
We will report also a closed product formula for the R-polynomials of permutations which are smaller than a transposition (i, j) under the Bruhat order.

We give now the result of [3], it is based on the enumerations of increasing subsequences of a permutation:

Theorem 1.20. *Let $u \in S_n$ and $w \in C_{i,j}(u)$ for some $i, j \in [n]$. Then*

$$\tilde{R}_{u,w^{-1}u}(t) = t^{k(w)-1}(t^2 + 1)^{\frac{1}{2}(d-k(w)+1)}$$

where $C_{i,j}(u) \stackrel{\text{def}}{=} \{(u(i_1), u(i_2), \ldots, u(i_s)) : s \in [n], i = i_1 < i_2 < \cdots < i_s = j$ and $u(i_1) < u(i_2) < \cdots < u(i_s)\}$, $d \stackrel{\text{def}}{=} inv(wu) - inv(u)$ and $k(w)$ is the length of the cycle w.

See [3], Sect.4.
On the following theorem will be based the definition of reducible class of permutations, to introduce the first theorem, we need to recall the definition of *restriction of a permutation.*
Let $u \in S_n$ and $i, j \in [n]$, $i \leq j$. We define the *restriction of* u *to* $[i, j]$ to be the unique permutation $u_{[i,j]} \in S([i, j])$ such that

$$u^{-1}(u_{[i,j]}(i)) < u^{-1}(u_{[i,j]}(i+1)) < \cdots < u^{-1}(u_{[i,j]}(j))$$

In other words to find the restriction of a permutation u to $[i, j]$ we can write u^{-1} and consider the set $\{u^{-1\, j,i}, u^{-1\, j,i+1}, \ldots, u^{-1\, j,j}\}_< = \{u^{-1}(i), u^{-1}(i+1) \ldots u^{-1}(j)\}$, then $u_{[i,j]} = [u(u^{-1\, j,i}), u(u^{-1\, j,i+1}), \ldots, u(u^{-1\, j,j})]$.

For example, consider the permutation $u = 4267351$ then $u_{[1,5]} = 42351$ since $u^{-1} = 7251634$ it results $\{u^{-1\, 5,1}, u^{-1\, 5,2}, \ldots, u^{-1\, 5,5}\}_< = \{1, 2, 5, 6, 7\}$, hence $u_{[1,5]}(1) = 4$, $u_{[1,5]}(2) = 2$, $u_{[1,5]}(3) = 3$, $u_{[1,5]}(4) = 5$, $u_{[1,5]}(5) = 1$.

Theorem 1.21. *Let $u, v \in S_n, u \leq v$. Suppose that there exist $1 = i_0 \leq i_1 < i_2 < \cdots < i_k \leq i_{k+1} = n$ such that $u^{-1}([i_j, i_{j+1}]) = v^{-1}([i_j, i_{j+1}])$, $\forall j = 0, \ldots, k$. Then*

$$\tilde{R}_{u,v}(t) = \prod_{j=0}^{k} \tilde{R}_{u([i_j,i_{j+1}]),v([i_j,i_{j+1}])}(t) \tag{1.1}$$

Finally we end with a result of different nature, which gives a closed product formula for permutations smaller than $(1, n)$:

Theorem 1.22. *Let $u, v \in S_n$ be such that $u \leq v \leq (i, j)$ for some $i, j \in [n]$, $i \neq j$. Then*

$$\tilde{R}_{u,v}(t) = t^a (t^2 + 1)^{inv(v) - inv(u) - a/2},$$

for some $a \in \mathbf{N}$

[See [16], corollary 4.2].

The exponent a that appears in the last formula, is related to reduced expressions for u and v, as is stated in the next theorem, where if $\tilde{u}$ is a reduced expression of a permutation u, then $\tilde{u}(s)$ is the number of times that s appears in $\tilde{u}$, for $s \in S = \{s_1, s_2, \ldots, s_{n-1}\}$, with $s_t = (t, t+1)$.

Theorem 1.23. *If $\tilde{v}$ is a fixed reduced expression of v that is a sub word of $s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_1$ and $\tilde{u}$ is a fixed reduced expression of u that is a sub word of $\tilde{v}$, then the following formula for the exponent a of Theorem 1.22, holds:*

$$a = \sum_{i=1}^{n} a_i$$

where

$$a_i = \begin{cases} 0, & \text{if } \tilde{v}(s_i) = 2,\ \tilde{u}(s_i) = 0 \\ & \text{and } s_i \text{ commutes with every } s_j,\ j > i \text{ such that } \tilde{u}(s_j) \neq 0 \\ \tilde{v}(s_i) - \tilde{u}(s_i), & \text{otherwise} \end{cases}$$

See [16], Theorem 4.1.

As final remark we say that there are classes of permutations for which are known particular characterizations, if not properly a closed product formula, which allow an easier calculation also in terms of computational time. As an example we give here a relationship between the R-polynomial and the q-analogue of Fibonacci numbers. We recall that for $n \in \mathbb{N}$ the n-Fibonacci polynomial is the polynomial $F(n) = F(n-1) + qF(n-2)$, where $F(0) \stackrel{\text{def}}{=} 1$, and $F(1) \stackrel{\text{def}}{=} 1$, see e.g. [6].

Theorem 1.24. *Let $n > 1$. Then*

$$\tilde{R}_{e,34\ldots n12}(t) = t^{2(n-2)} F(n-2)_{|t^{-2}}$$

where $F(n-2)_{|t^{-2}}$ is the $(n-2)$-Fibonacci polynomial in t^{-2}.

This has been proved in [17], Sect. 4.

Chapter 2

Conjugacy classes and R-polynomials

In this chapter we introduce an equivalence relation on $W \times W$ where W is a finite Coxeter group which preserves the R-polynomials and it implies the conjugacy relation in a sense that we will explain in Section 2 of this chapter. This relation allow us to consider the R-polynomial of a class and in the case of the symmetric group to investigate the relations between the cycle-type and the R-polynomials. What we will show in the next chapters is that "the R-polynomials split the conjugacy classes".

2.1 The equivalence relation

Definition 2.1. *Let W a finite Coxeter group with set of generators S. Let $u, v, x, y \in W$ such that $u \leq v$ and $x \leq y$.*

We say that $(u,v)R(x,y)$ if and only if one of the following holds:

1. $\exists\, s \in S$ *such that* $s \in D(u) \cap D(v)$ *and* $x = us$, $y = vs$
2. $\exists\, s \in S$ *such that* $s \notin D(u), s \notin D(v)$ *and* $x = us$, $y = vs$
3. $y = uw_0$ *and* $x = vw_0$
4. $\exists\, s \in S$ *such that* $s \in D_L(u) \cap D_L(v)$ *and* $x = su$, $y = sv$
5. $\exists\, s \in S$ *such that* $s \notin D_L(u), s \notin D_L(v)$ *and* $x = su$, $y = sv$
6. $y = w_0u$, $x = w_0v$

Let $\sim$ be the transitive closure of R

Proposition 2.2. *The relation $\sim$ is an equivalence relation on $W \times W$.*

Proof. We have only to prove that the relation R is symmetric.
Let $(u,v)R(x,y)$ and we consider the different cases given by the definition.

1. $(u,v)R(x,y)$ by 1) then $\exists s \in D(u) \cap D(v)$ such that $(x,y) = (us, vs)$. By definition of right descent $s \in D(u) \Leftrightarrow \ell(u) > \ell(us)$ and similarly $\ell(v) > \ell(vs)$. It is obvious that, for the same reason, s is not a descent of the permutations x and y and since $xs = u$ and $ys = v$ we have that $(x,y)R(u,v)$ by 2).

2. $(u,v)R(x,y)$ by 2) then $s \notin D(u), s \notin D(v)$ and $x = us$, $y = vs$. This implies that $\ell(u) < \ell(us) = \ell(x)$ and $\ell(v) < \ell(vs) = \ell(y)$, so $s \in D(x) \cap D(y)$, and being $u = xs$ and $v = ys$ we conclude that $(x,y)R(u,v)$, by 1).

3. $(u,v)R(x,y)$ by 3) then $y = uw_0$ and $x = vw_0$. We observe that $(x,y)R(yw_0, xw_0)$, and using the fact that $(w_0)^2 = e$ we have that $yw_0 = uw_0w_0 = u$, $xw_0 = vw_0w_0 = v$ from which follows that $(x,y)R(u,v)$.

4. $(u,v)R(x,y)$ by 4) then $\exists\ s \in S$ such that $s \in D_L(u) \cap D_L(v)$ and $x = su$, $y = sv$. By definition of left descent we have $\ell(u) > \ell(su)$ and $\ell(v) > \ell(sv)$ therefore $s \notin D_L(su), s \notin D_L(sv)$ and since $u = sx$, $v = sy$ we conclude that $(x,y)R(u,v)$ by 5).

5. $(u,v)R(x,y)$ by 5) then $\exists\ s \in S$ such that $s \notin D_L(u), s \notin D_L(v)$ and $x = su$, $y = sv$ and in as we did for the right version we have that $(x,y)R(u,v)$ by 4).

6. $(u,v)R(x,y)$ by 6) then $y = w_0u$, $x = w_0v$, as for the proof of 3) we observe $(x,y) = (w_0v, w_0u)R(w_0w_0u, w_0w_0v) = (u,v)$.

$\square$

We illustrate how the equivalence relation works in the next example on the symmetric group. The same procedure will be used to prove the main results of this thesis.

Example Let $u = 17523648$ and $v = 87324651$

$$\begin{aligned}
&u = 17523648 \xrightarrow{*w_0} 15\mathbf{6}42378 \xrightarrow{(3,4)} 154\mathbf{6}2378 \xrightarrow{(4,5)}\\
&v = 87324651 \phantom{\xrightarrow{*w_0}} 84\mathbf{6}32571 \phantom{\xrightarrow{(3,4)}} 843\mathbf{6}2571\\
&\quad 1542\mathbf{6}378 \xrightarrow{(5,6)} 15\mathbf{4}23678 \xrightarrow{(3,4)} 1\mathbf{5}243678 \xrightarrow{(2,3)}\\
&\quad 8432\mathbf{6}571 \phantom{\xrightarrow{(5,6)}} 84\mathbf{3}25671 \phantom{\xrightarrow{(3,4)}} 8\mathbf{4}235671\\
&\quad 12\mathbf{5}43678 \xrightarrow{(3,4)} 12453678 \xrightarrow{*^{-1}} 12534678 = x\\
&\quad 82\mathbf{4}35671 \phantom{\xrightarrow{(3,4)}} 82345671 \phantom{\xrightarrow{*^{-1}}} 82345671 = y
\end{aligned}$$

Therefore

$$(17523648, 87324651) \sim (12534678, 82345671)$$
$$R_{u,v}(q) = (q^2 - q + 1)^3 (q-1)^5 = R_{x,y}(q)$$

Properties of the equivalence relation

Proposition 2.3. *Let $u, v, x, y \in W$ such that $(u, v) \sim (x, y)$ then $R_{u,v}(q) = R_{x,y}(q)$.*

Proof. From Proposition 1.15, Theorem 1.12 and its left version 1.14 follows that if $(u, v) \sim (x, y)$ the associated R-polynomial are equals. In fact $(u, v) \sim (x, y)$ if there is a sequence of operations, the ones allowed by the definition of the equivalence relation, which transforms the pair (u, v) in the pair (x, y). Each operation preserves the R-polynomial as consequence of Proposition 1.15 and Theorem 1.12.
We spend a few words for the case 2), (the case 5) is analogous): $(u, v)R(x, y)$ by 2) then $s \notin D(u), s \notin D(v)$ and $x = us$, $y = vs$.We apply Theorem 1.12 to the pair (x, y) since $s \in D(x) \cap D(y)$, and the thesis follows.

□

The previous proposition implies that we can speak about the R-polynomial of a class, and it can be indicated as $R_{[u,v]}(q)$.

Now we prove that, if W is a Weyl group (see Definition 1.1), also the conjugacy class of uv^{-1} is preserved for each class $[u, v]$; nevertheless for each fixed conjugacy class, we might have different corresponding equivalence classes, which means that in each conjugacy class "we can find different R-polynomials", see section 2.3.

To prove the relation between $\sim$ and the conjugacy relation we need the next deep result due to Carter.

Theorem 2.4. *Every element in a Weyl group W is conjugate to its inverse.*

This is [7], Corollary of Theorem C.

We prove the

Theorem 2.5. *Let $u, v, x, y \in W$, such that $u \leq v$ and $x \leq y$. Then*

$$(u, v) \sim (x, y) \Rightarrow [u^{-1}v]_c = [x^{-1}y]_c$$

where $[\sigma]_c$ is the conjugacy class of $\sigma \in W$.

Proof. It is an easy calculation.

- If (u, v) and (x, y) are related by 1), 2) then $u = xs$, and $v = ys$; $x^{-1}y = (us)^{-1}(vs) = su^{-1}vs$ and since $s = s^{-1}$ then $[uv^{-1}]_c = [x^{-1}y]_c$

- If (u,v) and (x,y) are related by 4), 5) then $x = su$, $y = sv$ and $x^{-1}y = u^{-1}ssv = u^{-1}v$.

- If (u,v) and (x,y) are related by 3) we have that $x^{-1}y = w_0 v^{-1} u w_0$, so $[x^{-1}y]_c = [(uv^{-1})^{-1}]_c$ and by Theorem 2.4 we have the assert. The case 6) is similar to the last 1).

$\square$

We will indicate the equivalence class of a pair (u,v) by $[(u,v)]$.

Remarks The converse of the theorem is not true as it is showed in the next example, where $W = S_n$
Let $u = 123465$, $v = 563421$ then

$$R_{123465,563421}(q) = q^{10} - 3q^9 + 4q^8 - 2q^7 + q^5 - 2q^3 + 4q^2 - 3q + 1)(q-1)^2$$

and $u^{-1}v = (1,6)(2,5)$
Consider now $x = 153426$ and $y = 623451$, we observe that $x^{-1}y = (1,6)(2,5)$. If were $(x,y) \sim (u,v)$ then by Proposition 2.3 it must be $R_{u,v}(q) = R_{x,y}(q)$, but this is not true since $R_{153426,623451}(q) = (q-1)^4$.

We end this section observing that every statement given on R-polynomials is valid for $\tilde{R}$-polynomials and since we usually work with this last family we write as a corollary that

Corollary 2.6. *Let $u,v,x,y \in W$ such that $(u,v) \sim (x,y)$ then $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t)$.*

2.2 Irreducible classes of permutations

From now on we consider $W = S_n$.

We introduce a classification of the equivalence classes given in Section 1 of this chapter.
The concept of irreducible classes comes from Theorem 1.21. This result implies that it is possible, for certain pairs of permutations, to obtain an explicit formula for their R-polynomial by decomposing it into a product of non trivial R-polynomials for which it is known an explicit factorization.

Definition 2.7. *A pair (u,v) of permutations is said to be irreducible if the $\tilde{R}_{u,v}(q)$ satisfies the (1.1) only with the condition that there exists a unique $j \in [0, n+1]$ such that $\tilde{R}_{u_{(i_j,i_{j+1}]},v_{(i_j,i_{j+1}]}}(t) \neq 1$, i.e only one factor of the product is non trivial. Otherwise we say that (u,v) is a reducible pair. The equivalence class of a pair $[(u,v)]$ is said to be reducible or irreducible depending on the reducibility or not of (u,v).*

Example The pairs of permutations of the following Corollary 2.8 and Proposition 2.11 are irreducibles, as well as the ones of the following chapter.
This definition allow us to concentrate our attention on the irreducible pairs and consider Theorem 1.21 as an explicit formulas generating machinery. In fact from this theorem follows many other explicit formulas as the one Corollaries 2.13 and 2.15 in which are given explicit formulas for two reducible classes; these are obtained by product of R-polynomials associated to irreducible classes.

It is well known that in S_n, conjugacy classes are indexed by partitions, we recall here the notation that we adopt:
we say that a permutation $\sigma \in S_n$ is of type $1^{t_1}2^{t_2}\dots n^{t_n}$ if $\forall j \in [n]$, t_j is the number of cycles of σ of length j. Note that $n = \sum jt_j$, see [21], sect 1.3. Then the associated partition is $(n^{t_n}, \dots, 2^{t_2}, 1^{t_1})$, where i^{t_i} means that i is repeated t_i times.

The R-polynomials of the class $(2, 1, 1, \dots)$ are given in the next

Corollary 2.8. (2,1,1,...) *Let $u \in S_n$ such that $u(i) < u(j)$ for some $i, j \in [n]$, $i < j$. If $v = u(i, j)$, then*

$$\tilde{R}_{u,v}(t) = t(t^2 + 1)^{\frac{1}{2}(inv(v) - inv(u) - 1)}.$$

Proof. It is an easy consequence of Theorem 1.20 since $(i, j) \in C_{i,j}(u)$. □

From Corollary 2.8 and Theorem 1.16, we have immediately:

Corollary 2.9. *Let $u \in S_n$ such that $u(i) < u(j)$ for some $i, j \in [n]$, $i < j$. If $v = u(i, j)$, then*

$$R_{u,v}(q) = (q - 1)(q^2 - q - 1)^{\frac{1}{2}(inv(v) - inv(u) - 1)}.$$

From Theorem 1.20 we can derive the closed product formula for the partition $(3, 1, 1, \dots)$. Before giving it, we prove a technical lemma which is a consequence of Theorem 1.17. We will apply it in the proof of Proposition 2.11 and in the following chapters and sections.

Lemma 2.10. *Let $u, v \in S_n$, $u \le v$ be such that $(k, l) \in Inv(u) \cap Inv(v)$ and $u(i) = v(i), \forall i \in [k + 1, l - 1]$.
Suppose that $min\{u(k), v(k)\} \ge max\{u(l), v(l)\}$. Then there exists an index $r \in [k, l - 1]$ such that:*

$$\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t)$$

where $x = u\sigma$, $y = v\sigma$ and $\sigma = (k, k + 1, k + 2, \dots, r - 1, r, l, l - 1, l - 2, \dots, r + 1)$.

Proof. We have:

$$\begin{aligned} u &= u(1)\dots \boxed{u(k)}\, u(k+1)\dots u(l-1)\,\boxed{u(l)}\dots u(n) \\ v &= v(1)\dots \boxed{v(k)}\, u(k+1)\dots u(l-1)\,\boxed{v(l)}\dots v(n) \end{aligned}$$

We can suppose that $u(k+1) < \cdots < u(l-1)$ by Theorem 1.17.

Let $U \stackrel{\text{def}}{=} max\{v(l), u(l)\}$ and

$$r \stackrel{\text{def}}{=} max\{m \in [k+1, l-1] : u(m) < U\}$$

Then $u(k+1) < \cdots < u(r) < U < u(r+1) < \cdots < u(l-1)$ and the result follows by applying Theorem 1.17.

□

We say that an inversion which satisfies the hypotheses of Lemma 2.10 "behaves" as descent in the sense of Theorem 1.17.

We examine the situation in which $v \in S_n$ is obtained from a permutation u by applying a 3-cycle, there are several possibilities to consider. We write only the positions in which u and v are different, i.e. the ones that we "rotate" in u to find v, and we indicate with "..." the others.

Proposition 2.11. (3,1,1,...)
Let $u \in S_n$, $a, b, c \in [n]$, be such that $a < b < c$.
If (u, v) is one of the next pairs:

$$u = \dots\mathbf{a}\dots\mathbf{b}\dots\mathbf{c}\dots \quad \textit{and} \quad v = \dots\mathbf{c}\dots\mathbf{a}\dots\mathbf{b}\dots$$

$$u = \dots\mathbf{a}\dots\mathbf{b}\dots\mathbf{c}\dots \quad \textit{and} \quad v = \dots\mathbf{b}\dots\mathbf{c}\dots\mathbf{a}\dots$$

$$u = \dots\mathbf{a}\dots\mathbf{c}\dots\mathbf{b}\dots \quad \textit{and} \quad v = \dots\mathbf{c}\dots\mathbf{b}\dots\mathbf{a}\dots$$

$$u = \dots\mathbf{b}\dots\mathbf{a}\dots\mathbf{c}\dots \quad \textit{and} \quad v = \dots\mathbf{c}\dots\mathbf{b}\dots\mathbf{a}\dots$$

Then

$$\tilde{R}_{u,v}(t) = t^2(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

Moreover in the other cases, $\tilde{R}_{u,v}(t) = 0$, since $u \not\leq v$.

Proof. If $u = \dots a\dots b\dots c\dots$ this means that $w = (a, b, c) \in C_{a,c}(u)$; when $v = \dots c\dots a\dots b\dots$ we can apply Theorem 1.20 since $v = w^{-1}u$. This implies that $\tilde{R}_{u,v}(t) = t^2(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$ being $k(w) = 3$.
If $v = \dots b\dots c\dots a \cdots = (a, b, c)u$ then we cannot apply directly the same result. By Proposition 1.18 the pair (vw_0, uw_0) has the same R-polynomial of the pair (u, v).

We have $\begin{array}{l} vw_0 = \\ uw_0 = \end{array} \cdots \begin{array}{c} a \\ c \end{array} \cdots \boxed{\begin{array}{c} c \\ b \end{array}} \cdots \boxed{\begin{array}{c} b \\ a \end{array}} \cdots$ and if we define $i = (vw_0)^{-1}(c) = (uw_0)^{-1}(b)$, $j = (vw_0)^{-1}(b) = (uw_0)^{-1}(a)$ then $(i,j) \in Inv(uw_0) \cap Inv(vw_0)$. We can apply Lemma 2.10, ((i,j) behaves as a descent), so that $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t)$, with

$$\begin{array}{l} x = \\ y = \end{array} \cdots \begin{array}{c} a \\ c \end{array} \cdots \begin{array}{c} i_1 \\ i_1 \end{array} \cdots \begin{array}{c} i_k\, b\, c\, i_{k+1} \\ i_k\, a\, b\, i_{k+1} \end{array} \cdots \begin{array}{c} i_s \\ i_s \end{array} \cdots,$$

which satisfies the conditions of Theorem 1.20.

We consider now $u = \ldots a \ldots c \ldots b \ldots$

If $v = \ldots c \ldots b \ldots a \cdots = (a,c,b)u$ then the pair (u,v) is exactly the pair that we called (vw_0, uw_0) in the previous case and we are done.

If $v = \ldots b \ldots a \ldots c \cdots = (a,b,c)u$, by Theorem 1.9 $u \not\leq v$ and $\tilde{R}_{u,v}(t) = 0$.

Finally if $u = \ldots b \ldots a \ldots c \ldots$ and $v = \ldots c \ldots b \ldots a \cdots = (a,c,b)u$, we observe that $vw_0 = \ldots a \ldots b \ldots c \ldots$ and $uw_0 = \ldots c \ldots a \ldots b \ldots$ and we can apply Theorem 1.20 as we have already seen. If $v = (a,b,c)u = \ldots a \ldots c \ldots b \ldots$ it is obvious that $u \not\leq v$.

The remaining permutations u to consider are the ones in which c is in the first position, but this implies that for each 3-cycle that we apply to u we obtain a permutation v which is not greater than u, so we have completed the proof.

□

Corollary 2.12. *Let $u, v \in S_n$ be under the conditions of Proposition 2.11. Then*

$$R_{u,v}(q) = (q-1)^2(q^2-q+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

The pairs of the next theorems are reducible, in fact they are obtained from Theorem 1.21 and the formulas of the previous corollary and proposition. This is the first example of this thesis of the fact that "explicit formulas can produce other explicit formulas".

Corollary 2.13. Two disjoint transpositions *Let $u, v \in S_n$ such that $v = u(i,j)(k,l)$ and $i < j < k < l$. If $u < v$, we have that*

$$\tilde{R}_{u,v}(t) = t^2(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

Proof. We consider pairs of permutations (u,v) such that $v = u(i,j)(k,l)$ and $i < j < k < l$, i.e. (k,l) and (i,j) are disjoint transpositions. We write a pair of this type as follows:

$$\begin{array}{l} u = \\ v = \end{array} \cdots \begin{array}{c} u(i) \\ u(j) \end{array} \cdots \begin{array}{c} u(j) \\ u(i) \end{array} \cdots \begin{array}{c} u(k) \\ u(l) \end{array} \cdots \begin{array}{c} u(l) \\ u(k) \end{array} \cdots.$$

The possible orderings of $u(i), u(j), u(k), u(l)$ consistent with Theorem 1.9 are: $u(i) < u(j) < u(k) < u(l)$, $u(i) < u(k) < u(j) < u(l)$, $u(i) < u(k) < u(l) < u(j)$. There are also the orderings symmetric to these among the multiplication by w_0: $u(k) < u(l) < u(i) < u(j)$, $u(k) < u(i) < u(l) < u(j)$, $u(k) < u(i) < u(j) < u(l)$. It is clear that the fundamental condition for being $u < v$ is that $u(i) < u(j)$ and $u(k) < u(l)$ as in the case of 2-cycles. We suppose to be under one of the previous ordering for $u(i), u(j), u(k), u(l)$. We apply Theorem 1.21 on $x = u^{-1}$ and $y = v^{-1}$. It is obvious that $x^{-1}[1, j] = y^{-1}[1, j]$ and $x^{-1}[j + 1, n] = y^{-1}[j + 1, n]$ Then $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t) = \tilde{R}_{x_{[1,j]},y_{[1,j]}}(t) \times \tilde{R}_{x_{[j+1,n]},y_{[j+1,n]}}(t)$ In a intuitive sense we have broken x and y in two pairs of permutations in which the second permutation is obtained by the first applying a transposition; equivalently we have that $\tilde{R}_{u,v}(t)$ is the product of two polynomials of type $= t(t^2 + 1)^{\frac{1}{2}(d-1)}$. ☐

Corollary 2.14. *Let $u, v \in S_n$ such that $v = u(i, j)(k, l)$ and $i < j < k < l$. If $u < v$, we have that*

$$R_{u,v}(q) = (q - 1)^2(q^2 - q + 1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

To prove the following result we use the same method.

Proposition 2.15. A 3-cycle and a disjoint transposition *Let $u, v \in S_n$ be such that $v = u(i, j)(a, b, c)$ and $i < j < a < b < c$. If $u < v$, then*

$$\tilde{R}_{u,v}(t) = t^3(t^2 + 1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

Proof. In Proposition 2.11 we analyzed the possibilities for the relative order of $u(a)$, $u(b)$, $u(c)$, to be $u < v$ when $v = u\tau$, and τ is the 3-cycle (a, b, c) or its inverse. (Note that in 2.11 we used a different notation: a,b,c indicated the values of three fixed elements and not the positions as in this case). Here we refer to that result and assume to be under one of those conditions and to add the hypothesis that $u(i) < u(j)$. As before we apply Theorem 1.21 to $x = u^{-1}$ and $y = v^{-1}$ since $x^{-1}[1, j] = y^{-1}[1, j]$ and $x^{-1}[j + 1, n] = y^{-1}[j + 1, n]$.
Then $\tilde{R}_{u,v}(t) = \tilde{R}_{x,y}(t) = \tilde{R}_{x_{[1,j]},y_{[1,j]}}(t) \times \tilde{R}_{x_{[j+1,n]},y_{[j+1,n]}}(t)$. Now we have $\tilde{R}_{x_{[1,j]},y_{[1,j]}}(t) = t(t^2 + 1)^{\frac{1}{2}(d-1)}$, while, by Proposition 2.11 the other factor is now $\tilde{R}_{x_{[j+1,n]},y_{[j+1,n]}}(t) = t^2(t^2 + 1)^{\frac{1}{2}(d_1-1)}$, where $d_1 = inv(y_{[j+1,n]}) - inv(x_{[j+1,n]})$. ☐

Corollary 2.16. *Let $u, v \in S_n$ such that $v = u(i, j)(a, b, c)$ and $i < j < a < b < c$. If $u < v$, we have that*

$$R_{u,v}(q) = (q - 1)^3(q^2 - q + 1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$$

It is clear that in a similar way we can compute the R-polynomials of pairs (u, v), $u < v$, in which v is obtained from u by applying k disjoint transpositions or mixed 3-cycles and 2-cycles but all disjoints, we do not write explicitly these formulas.

2.3 Cycle-type under the transposition $(1, n)$

Proposition 2.17. *For every conjugacy class in S_n there is at least one representative which is less than $(1, n)$ under the Bruhat order.*

Proof. We say that a permutation $\sigma \in S_n$ is of type $1^{t_1}2^{t_2}\dots n^{t_n}$ if $\forall j \in [n]$, t_j is the number of cycles of σ of length j. Note that $n = \sum jt_j$, see [21], sect 1.3.
Let $[\sigma]_c$ the conjugacy class of σ; we can associate to this class the corresponding cycle type.

We prove that for any fixed cycle-type $j_1^{t_{j_1}} j_2^{t_{j_2}} \dots j_k^{t_{j_k}}$ where $1 \leq j_1 < j_2 < \dots < j_k \leq n$ there exists a permutation $u < (1, n)$ of this type. We consider $u \neq e$.
We can write $(1, n) = s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2} \dots s_2 s_1$, and this is a reduced expression. We consider the permutation

$$
\begin{aligned}
u = & \underbrace{s_{a_1^{j_1}} s_{a_1^{j_1}-1} \dots s_{a_1^{j_1}-j_1+2}}_{j_1-cycle} \dots \underbrace{s_{a_{t_{j_1}}^{j_1}} s_{a_{t_{j_1}}^{j_1}-1} \dots s_{a_{t_{j_1}}^{j_1}-j_1+2}}_{j_1-cycle} \\
& \underbrace{s_{a_1^{j_2}} s_{a_1^{j_2}-1} \dots s_{a_1^{j_2}-j_2+2}}_{j_2-cycle} \dots \underbrace{s_{a_{t_{j_2}}^{j_2}} s_{a_{t_{j_2}}^{j_2}-1} \dots s_{a_{t_{j_2}}^{j_2}-j_2+2}}_{j_2-cycle} \\
& \dots \\
& \underbrace{s_{a_1^{j_k}} s_{a_1^{j_k}-1} \dots s_{a_1^{j_k}-j_k+2}}_{j_k-cycle} \dots \underbrace{s_{a_{t_{j_k}}^{j_k}} s_{a_{t_{j_k}}^{j_k}-1} \dots s_{a_{t_{j_k}}^{j_k}-j_k+2}}_{j_k-cycle}
\end{aligned}
$$

together with the following conditions:

1. $a_i^{j_m} - j_m + 2 > a_{i+1}^{j_m} + 1$, $\forall i \in [j_m - 1]$ and $\forall m \in [k]$
2. $a_{t_{j_i}}^{j_i} - j_i + 2 > a_1^{j_{i+1}} + 1$, $\forall i \in [k-1]$
3. $a_1^{j_1} \leq n - 1$ and $a_{t_{j_k}}^{j_k} - j_k + 2 \geq 1$

The permutation u is of type $j_1^{t_{j_1}} j_2^{t_{j_2}} \dots j_k^{t_{j_k}}$ in fact we note that

$$
\begin{aligned}
& s_{a_i^{j_m}} s_{a_i^{j_m}-1} \dots s_{a_i^{j_m}-j_m+2} = \\
& = (a_i^{j_m}, a_i^{j_m}+1)(a_i^{j_m}-1, a_i^{j_m}) \dots (a_i^{j_m}-j_m-1, a_i^{j_m}-j_m) \\
& = (a_i^{j_m}-j_m-1, a_i^{j_m}+1, a_i^{j_m}, a_i^{j_m}-1, \dots, a_i^{j_m}-j_m)
\end{aligned}
$$

$\forall i \in [t_{j_m}]$ and $\forall m \in [k]$.

Moreover for any fixed $m \in [k]$ the condition (1) implies that the j_m-cycles are disjoint, so that in each rows of u there are t_{j_m} disjoint j_m-cycles; for the condition (2) also the cycles with different length, of u, are disjoint.

Since we have written u as subexpression of $(1, n)$, we are done, by Theorem 1.6.

As extremal case we have that an n-cycle corresponds to $a_1 = n-1$, $k = 1$ and $j_1 = n$, it is the permutation $s_{n-1}s_{n-2}\dots s_2 s_1 = (1, n, n-1, n-2, \dots, 3, 2)$.

We note that if u is a permutation which has a reduced decomposition of the described type then $u^{-1} < (1, n)$, since it has a reduced decomposition which is subexpression of $s_1 s_2 \dots s_{n-1}$.

□

In terms of equivalence classes and the associated $\tilde{R}$-polynomials we have the next result:

Corollary 2.18. *For very fixed cycle type $1^{t_1}2^{t_2}\dots n^{t_n}$ there exists a class of permutations $[(u, v)]$ such that $u^{-1}v$ is of type $1^{t_1}2^{t_2}\dots n^{t_n}$ and*

$$R_{[(u,v)]}(q) = (q-1)^a (q^2 - q + 1)^{\frac{inv(v)-inv(u)-a}{2}}$$

where the exponent a is given by Proposition 1.23.

Proof. From 2.17 we have that there exists a representative $v < (1, n)$ for any conjugacy class. This implies that $\tilde{R}_{e,v}(t)$ is a polynomial of the required type by Theorem 1.22 and the result follows by Theorem 1.16.

□

Example: the case of S_6

We illustrate the proof for $n = 6$:
$(1, 6) = s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1$.

$type(2, 1, 1, 1, 1) \rightarrow s_5, s_4, s_3, s_2, s_1$

$type(2, 2, 1, 1) \rightarrow s_5 s_3, s_4 s_2, s_3 s_1$

$type(2, 2, 2) \rightarrow s_5 s_3 s_1$

$type(3, 1, 1, 1) \rightarrow s_5 s_4, s_4 s_3, s_3 s_2, s_2 s_1$

$type(3, 2, 1) \rightarrow s_5 s_3 s_2, s_4 s_2 s_1, s_5 s_4 s_2, \dots$

$type(3, 3) \rightarrow s_5 s_4 s_2 s_1$

$type(4,1,1) \to s_4 s_3 s_2$

$type(4,2) \to s_5 s_3 s_2 s_1$

$type(5,1) \to s_4 s_3 s_2 s_1$

$type(6) \to s_5 s_4 s_3 s_2 s_1$

There is also an enumerative problem on the cardinality of the set of permutations which are under $(1,n)$. So far we have only computational results, in fact we have determined this cardinality up to $n=8$, by using Maple C.A.S.:

We define $U(n) \stackrel{\text{def}}{=} |\{u \in S_n : u < (1,n)\}|$, $\forall n \in \mathbb{N}, n \geq 2$.
For example:

$$U(4) = |\{[1,2,3,4],[1,2,4,3],[1,3,2,4],[1,3,4,2],[1,4,2,3],[1,4,3,2],[2,1,3,4],[2,1,4,3]$$
$$[3,2,4,1],[2,3,1,4],[2,3,4,1],[2,4,1,3],[2,4,3,1],[3,1,2,4],[3,1,4,2],[3,2,1,4],$$
$$[4,1,2,3],[4,1,3,2],[4,2,1,3],[4,2,3,1]\}| = 20$$

In the following table, we list the computed cardinalities:

n	**U(n)**
2	2
3	6
4	20
5	68
6	232
7	792
8	2704

Chapter 3

Explicit formulae for Kazhdan-Lusztig R-polynomials of some irreducible classes

This chapter contains our original results, these are the explicit formulas for the case in which the considered pairs are related to conjugacy classes of a k-cycle nested to a transposition. They both follow the idea that in a pair (u, v) the second elements is obtained multiplying u, in which occurs a fixed pattern, by a particular cycle which put the values of the pattern in their natural ordering in v. This requires that the element that we move in u are not in increasing order. This idea is different from the one of the increasing subsequences introduced by Brenti in [3] and of which are consequence the results contained in Chapter 2.

3.1 R-polynomials of permutations containing a pattern $k123\ldots k-1$ or $23\ldots(k-1)k1$

To express the properties which characterize the families of permutations that are object of this section we use the notion of pattern in a permutation, that we recall in the next:

Definition 3.1. *A permutation $w = w_1w_2\ldots w_n \in S_n$* **contains a pattern** *$v \in S_k$, $k \leq n$, if there exists a sequence $w_{i_1}w_{i_2}\ldots w_{i_k}$ with the same relative order as $v = v_1v_2\ldots v_k$; usually we write $v_1v_2\ldots v_k$ a pattern v in w.*

See, e.g [22].
For example the permutation $w = 7245163$ contains a pattern 1234 which is 2456.

We say that a permutation w *contains a pattern* v *between two fixed positions* $i,\ j$, if there exists a sequence $w_{i_1}w_{i_2}\dots w_{i_k}$ with the same relative order as $v = v_1v_2\dots v_k$ and $i < i_1 < i_2 < \dots i_k < j$
For example, $w = 12\mathbf{653}47$ contains a pattern 321 between 1 and 7.

Lemma 3.2. *Let $u \in S_n$ and $1 \leq i < a_1 < a_2 < a_3 < \cdots < a_k < j \leq n$, such that one of the following conditions holds:*

- *u contains a pattern $k123\dots k-1$ at the positions $a_1,\ a_2,\ \dots,\ a_k$ and $v = u(i,j)(a_1, a_2, a_3, \dots, a_k)$*
- *u contains a pattern $23\dots(k-1)k1$ at the positions $a_1,\ a_2,\ \dots,\ a_k$ and $v = u(i,j)(a_1, a_k, a_{k-1}, \dots, a_2)$*

Then $u < v \iff u(i) < min\{u(a_s) : s \in [k]\}$ and $u(j) > max\{u(a_s) : s \in [k]\}$.

Proof. We prove the statement by using Theorem 1.9. We can suppose $u < v$.
In both cases immediately we have that $u(i) < u(j)$; in fact if it were $u(j) < u(i)$, considering the i-th rearrangement of

$$\{u(1), u(2), \dots, u(i)\} = \{u^{i,1}, u^{i,2}, \dots, u^{i,s}, u(i), u^{i,s+2} \dots\}_{\leq},$$

where $u^{i,s+1}$ is the position occupied by $u(i)$ in the i-th rearrangement, and of $\{v(1) = u(1), v(2) = u(2), \dots, v(i) = u(j)\} = \{u^{i,1}, u^{i,2}, \dots, u(j), u^{i,s}, u^{s+2,i}, \dots\}_{\leq}$, under the notation of Theorem 1.9, we will have $u(i) = u^{i,s+1} \nleq v^{i,s+1} = u_{i,s}$, which implies $u \nleq v$, a contradiction.
We prove the part 1) then we have $u(a_2) = min\{u(a_s)|s \in [k]\}$ and $max\{u(a_s)|s \in [k]\} = u(a_1) < u(j)$. For simplicity, we change the notation in this way: $a_1 = a$ and $a_2 = b$. The two permutations can be written as follows:

$$\begin{aligned} u &= 1\ u(2)\dots u(a-1)u(a)u(a+1)\ \dots u(b-1)u(b)u(b+1)\ \dots\dots \\ v &= n\ u(2)\dots u(a-1)u(b)u(a+1)\dots u(b-1)u(a_3)u(b+1)\dots\dots \end{aligned}$$

$$\begin{matrix} u(a_{k-1}-1)u(a_{k-1})u(a_{k-1}+1)\dots u(a_k-1)u(a_k)u(a_k+1)\dots u(n-1)\ n \\ u(a_{k-1}-1)u(a_k)u(a_{k-1}+1)\quad \dots u(a_k-1)u(a)u(a_k+1)\dots u(n-1)\ \ 1 \end{matrix}$$

We have to show that $u(i) < u(b)$ and that $u(a) < u(j)$. Suppose that $u(a) < u(i)$, the a-th rearrangements will be:

$$\{u(1), \dots, u(i), \dots, u(a)\} = \{u^{a,1}, \dots, u^{a,t}, \mathbf{u}(\mathbf{a}), u^{a,t+2}, \dots, u(i), \dots\}_{\leq}$$

$$\{u(1), \dots, u(j), \dots u(b)\} = \{u^{a,1}, \dots, u(b), \dots, \mathbf{u^{a,t}}, u^{a,t+2}, \dots, u(j), \dots\}_{\leq},$$

where $u^{m,t+1}$ is the position occupied by $u(a)$ in the m-th rearrangement. It follows that $u \not\leq v$, since $u(a) = u^{a,t+1} \not\leq v^{a,t+1} = u^{a,t}$.
To conclude this part suppose now that $u(b) < u(i) < u(a)$ and consider again the a-th rearrangement of u and v:

$$\{u^{a,1}, \ldots, u^{a,s-1}, \mathbf{u(i)}, u^{a,s+1}, \ldots, u^{a,t}, u(a), u^{a,t+2}, \ldots\}_{\leq}$$

$$\{u^{a,1}, \ldots, u(b), \ldots, \mathbf{u^{a,s-1}}, u^{a,s+1}, \ldots, u^{a,t}, u^{a,t+2}, \ldots, u(j), \ldots\}_{\leq}$$

We have again a contradiction because $u^{a,s} = u(i) \not\leq v^{a,s} = u^{a,s-1}$; moreover from the same rearrangement we can conclude that $u(a) < u(j)$, in fact if $u(a) > u(j)$ we will have again $u(a) = u^{a,t+1} \not\leq v^{a,t+1} = u^{a,t}$.

To prove part 2) of this lemma we use the fact that the map $w \mapsto w^{-1}$ is an automorphism of Bruhat order, as stated in Proposition 1.10: $u \leq v \Longleftrightarrow u^{-1} \leq v^{-1}$. Now we have that $u(a_k) = min\{u(a_s)|s \in [k]\}$ and $u(a_k - 1) = max\{u(a_s)|s \in [k]\} = u(a_{k-1}) < u(j)$ and

$$\begin{aligned} u &= 1\, u(2) \ldots u(a_1 - 1)u(a_1)u(a_1 + 1) \ldots u(a_2 - 1)u(a_2)u(a_2 + 1) \ldots\ldots \\ v &= n\, u(2) \ldots u(a_1 - 1)u(a_k)u(a_1 + 1) \ldots u(a_2 - 1)u(a_1)u(a_2 + 1) \ldots\ldots \end{aligned}$$

$$\begin{aligned} &u(a_{k-1} - 1)u(a_{k-1})u(a_{k-1} + 1) \ \ldots u(a_k - 1)u(a_k)u(a_k + 1) \ldots u(n-1) \ \ n \\ &u(a_{k-1} - 1)u(a_{k-2})u(a_{k-1} + 1) \ldots u(a_k - 1)u(a_{k-1})u(a_k + 1) \ldots u(n-1)\ 1 \end{aligned}$$

Suppose that $u < v$ and for simplicity of notation define $u_s = u(s)$ for every $s \in [n]$, then $u^{-1}(u_s) = s$. The problem is to establish the natural ordering of the elements of the set $\{u_{a_{k-1}}, u_{a_k}, u_i, u_j\}$; from the hypothesis we now that $u_{a_k} < u_{a_1} < u_{a_2} < \cdots < u_{a_{k-1}}$ and, from the initial part of this proof, that $u_i < u_j$. Depending on this ordering is the position of the elements $i, a_1, a_2, a_3, \ldots, a_k, j$ on the inverse of u and consequently in v^{-1}:

$$\begin{aligned} u^{-1} &= \ldots a_k \ldots a_1 \ldots\ldots a_{k-2} \ldots a_{k-1} \ldots \\ v^{-1} &= \ldots a_1 \ldots a_2 \ldots\ldots a_{k-1} \ldots \ \ a_k \ \ \ldots \end{aligned}$$

and we want to determine the position of the column $\genfrac{}{}{0pt}{}{i}{j}$ and of the column $\genfrac{}{}{0pt}{}{j}{i}$. Since u^{-1} and v^{-1} are under the conditions of part 1), we can conclude by symmetry that $u_i < u_{a_k}$ and $u_{a_{k-1}} < u_j$, since the right positions of columns $\begin{matrix} i \\ j \end{matrix}$ and $\begin{matrix} j \\ i \end{matrix}$ are the following:

$$\begin{aligned} u^{-1} &= \ldots i \ldots a_k \ldots a_1 \ldots\ldots a_{k-2} \ldots a_{k-1} \ldots j \ldots \\ v^{-1} &= \ldots j \ldots a_1 \ldots a_2 \ldots\ldots a_{k-1} \ldots \ \ a_k \ \ \ldots i \ldots \end{aligned}$$

$\square$

Theorem 3.3. *Let $u \in S_n$ and $1 \leq i < a_1 < a_2 < a_3 < \cdots < a_k < j \leq n$, such that u contains a pattern $k123\ldots k-1$ at the positions a_1, a_2, $\ldots$, a_k.*
If $u < v = u(i,j)(a_1, a_2, a_3, \ldots, a_k)$ and $D(u) \cap D(v) = \emptyset = D_L(u) \cap D_L(v)$ then

$$(u, v) \sim (x, (i, j))$$

where x is a k-cycle, which is a pattern $k123\ldots k-1$.

Proof. From Lemma 3.2 we have that $u(i) < u(a_2)$ and that $u(a_1) < u(j)$.
We can assume, without loss of generality, that $i = 1$, $j = n$ and $u(1) = 1$, $u(n) = n$.
Moreover the hypothesis $D(u) \cap D(v) = \emptyset$ forces also that the values in the intervals $[a_m + 1, a_{m+1} - 1]$ are in increasing order, $\forall m \in [2, k-1]$.
We prove now that:

$$\{a_1\} \subseteq D(u) \subseteq \{a_1, a_2 - 1, a_3 - 1, \ldots, a_k - 1\}$$

$$\{1, n-1\} \subseteq D(v) \subseteq \{1, a_2, a_3, \ldots, a_{k-1}, n-1\}$$

The pair (u, v) can be represented as follows:

$$\begin{aligned} u &= 1\, u(2) \ldots u(a_1 - 1)\, u(a_1)\, u(a_1 + 1) \ldots u(a_2 - 1) u(a_2) u(a_2 + 1) \ldots\ldots \\ v &= n\, u(2) \ldots u(a_1 - 1)\, u(a_2)\, u(a_1 + 1) \ldots u(a_2 - 1) u(a_3) u(a_2 + 1) \ldots\ldots \end{aligned}$$

$$\begin{aligned} &u(a_{k-1} - 1)\, u(a_{k-1})\, u(a_{k-1} + 1) \ldots u(a_k - 1) u(a_k) u(a_k + 1) \ldots u(n-1)\, n \\ &u(a_{k-1} - 1)\quad u(a_k)\quad u(a_{k-1} + 1) \ldots u(a_k - 1) u(a_1) u(a_k + 1) \ldots u(n-1)\, 1 \end{aligned}$$

First of all we note that $a_1 \in D(u)$ because of the pattern, in fact if $u(a_1) < u(a_1 + 1)$, then being $u(a_2) < u(a_3) < u(a_1) < u(a_1 + 1) < \cdots < u(a_2 - 1)$ it will be $a_2 - 1 \in D(u) \cap D(v)$.
Concerning the other possible right descents we observe that $\forall m \in [2, k]$, $a_m - 1$ can be in $D(u)$ but not in $D(v)$ since $\forall m \in [2, k-1]$, $u(a_m) < v(a_m) = u(a_{m+1}) < u(a_m - 1)$ and $u(a_k) < v(a_k) < u(a_1)$; it is trivial that if $a_m - 1 \in D(v)$ then $a_m \in D(u)$.
Similarly, $\forall m \in [2, k-1]$, a_m can be in $D(v)$ but not in $D(u)$, in fact if $a_m \in D(u)$ it will be $u(a_m + 1) < u(a_m) < v(a_m)$, and then $a_m \in D(v)$.

From the assumption that $D_L(u) \cap D_L(v) = \emptyset$, it follows that $a_1 - 1, a_k \notin D(v)$, (clearly these descents cannot belong to $D(u)$, since this will imply immediately that they are common descents). In fact suppose that $(a_1 - 1) \in D(v)$, i.e. $u(a_1 - 1) > u(a_2)$, and define $t \stackrel{\text{def}}{=} max\{m \in [2, a_1 - 2] : u(m) < u(a_2)\}$.
It follows that $1 < u(2) < \cdots < u(t) < u(a_2) < u(t+1) < \cdots < u(a_1 - 1) < u(a_1)$ and therefore $u(m) = m$, $\forall m \in [2, t]$, $u(a_2) = t + 1$ and $u(t+1) = t + 2$. This implies that $u^{-1}(t+1) = a_2 > u^{-1}(t+2) = t + 1$ and $v^{-1}(t+1) = a_1 > v^{-1}(t+2) = t + 1$,

which is a contradiction, since this means that $(t+1,t+2) \in D_L(u) \cap D_L(v)$
Analogously, if $a_k \in D(v)$, i.e. $u(a_k+1) < u(a_1)$ and $t \stackrel{\text{def}}{=} max\{m \in [a_k+1, n-1] : u(m) < u(a_1)\}$, then the inequality chain $u(a_k+1) < \cdots < u(t) < u(a_1) < u(t+1) < \cdots < u(n-1)$ holds. We can conclude that $\forall m \in [a_k+1, t]$, $u(m) = m-1$, $u(a_1) = t$ while $u(m) = m$, $\forall m \in [t+1, n-1]$ and therefore that $(t-1,t) \in D_L(u) \cap D_L(v)$ since $u^{-1}(t-1) = t > u^{-1}(t) = a_1$ and $v^{-1}(t-1) = t > v^{-1}(t) = a_k+1$. These considerations enable us to take $a_1 = 2$, $a_k = n-1$, since $\forall m \in [2, a_1-2] \cup [a_k+1, n-1]$, $u(m) = m$.
Another consequence of $D_L(u) \cap D_L(v) = \emptyset$ is that $u(a_s-1) < u(a_s+1)$, $\forall s \in [2,k]$: the simplest case is when $a_s - 1 \notin D(u)$, since together with the fact that $a_s \notin D(u)$, it follows $u(a_s-1) < u(a_s) < u(a_s+1)$.
Now assume that $\exists \bar{s} \in [2,k]$ such that $a_{\bar{s}} - 1 \in D(u)$ and $u(a_{\bar{s}}-1) > u(a_{\bar{s}}+1)$; let $p = max\{m \in [a_{\bar{s}}+1, a_{\bar{s}+1}-1] : u(m) < u(a_{\bar{s}}-1)\}$, then $u(p) < u(a_{\bar{s}}-1) < u(p+1)$, more precisely if $u(a_{\bar{s}}-1) = t$ then $u(p_s+1) = t+1$, so that $u^{-1}(t) > u^{-1}(t+1)$ and since u and v do not differ in these positions then $(t,t+1) \in D_L(u) \cap D_L(v)$, which is a contradiction.

We prove the statement by induction on $|D(v)|$, so we first consider the minimal case for $D(v)$.

$\boldsymbol{D(v) = \{1, n-1\}}$

Trivially from this follows that $v = (1,n)$ and that $u = (2, n-1, a_{k-1}, a_{k-2}, \ldots, a_3, a_2)$.

$\boldsymbol{D(v) = \{1, a_2, a_3, \ldots, a_{k-1}, n-1\}}$ and $\boldsymbol{D(u) \subseteq \{2, a_2-1, a_3-1, \ldots, a_{k-1}-1, n-2\}}$

In order to determine the structure of u, which is a consequence of its descents set, we define for every $i \in [3,k]$ the set

$$A_i \stackrel{\text{def}}{=} \{m \in [a_{i-1}+1, a_i-1] : u(m) < u(a_i)\}.$$

We note that, for a fixed i, the set A_i, is not empty since $a_i \in D(v)$, in fact this implies that $u(a_i) > u(a_{i-1}+1)$ and then $a_{i-1}+1 \in A_i$. Let $M_i = maxA_i$, therefore we have $u(M_i) < u(a_i) < u(M_i+1)$.

The definition of M_i leads to the following inequalities:
$2 = u(a_2) < u(3) < \cdots < u(a_2-1) < u(a_2+1) < \cdots < u(M_3) < u(a_3) < u(M_3+1) < \cdots < u(a_3-1) < u(a_3+1) < \cdots < u(M_4) < u(a_4) < u(M_4+1) < \cdots < u(a_4-1) < u(a_4+1) < \cdots < \cdots < u(a_{k-2}-1) < u(a_{k-2}+1) < \cdots < u(M_{k-1}) < u(a_{k-1}) < u(M_{k-1}+1) < \cdots < u(a_{k-1}-1) < u(a_{k-1}+1) < \cdots < u(M_k) < u(n-1) < u(M_k+1) < \cdots < u(n-2) < u(2) = n-1$.
Therefore the structure of (u,v) is completely determined since:
$u(a_i) = M_i$, $\forall i \in [3,k]$, (remember that $a_k = n-1$), $u(h) = h$, $\forall h \in [3, a_2-1] \cup (\bigcup_{i=3}^{k} [M_i+1, a_i-1])$, while $u(h) = h-1$, $\forall h \in \bigcup_{i=3}^{k} [a_{i-1}+1, M_i]$.

We write explicitly (u,v):

$$\begin{array}{l} u = 1\; n-1 \\ v = n\quad 2 \end{array}\boxed{\begin{array}{c} 3\dots a_2-1 \\ 3\dots a_2-1 \end{array}}\begin{array}{c} 2 \\ M_3 \end{array}\boxed{\begin{array}{c} a_2\dots M_3-1\; M_3+1\dots a_3-1 \\ a_2\dots M_3-1\; M_3+1\dots a_3-1 \end{array}}\begin{array}{c} M_3 \\ M_4 \end{array}$$

$$\boxed{\begin{array}{c} a_3\dots M_4-1\; M_4+1\dots a_4-1 \\ a_3\dots M_4-1\; M_4+1\dots a_4-1 \end{array}}\begin{array}{c} M_4 \\ M_5 \end{array}\boxed{\begin{array}{c} a_4\dots M_5-1\; M_5+1\dots a_5-1 \\ a_4\dots M_5-1\; M_5+1\dots a_5-1 \end{array}}$$

$$\begin{array}{c} \dots\dots a_{k-2}-1\; M_{k-2} \\ \dots\dots a_{k-2}-1\; M_{k-1} \end{array}\boxed{\begin{array}{c} a_{k-2}\dots M_{k-1}-1\; M_{k-1}+1\dots a_{k-1}-1 \\ a_{k-2}\dots M_{k-1}-1\; M_{k-1}+1\dots a_{k-1}-1 \end{array}}$$

$$\begin{array}{c} M_{k-1} \\ M_k \end{array}\boxed{\begin{array}{c} a_{k-1}\dots M_k-1\; M_k+1\dots n-2 \\ a_{k-1}\dots M_k-1\; M_k+1\dots n-2 \end{array}}\begin{array}{c} M_k\quad n \\ n-1\;1 \end{array}$$

Since we have assumed that $D(u)\subseteq\{2,a_2-1,a_3-1,\dots,a_{k-1}-1,n-2\}$, we observe that if $a_i-1\notin D(u)$ then $A_i=[a_{i-1}+1,a_i-1]$ and then $M_i=a_i-1$, $u(M_i)=a_i-2$ and $u(a_i)=a_i-1$.
Therefore the block $\boxed{\begin{array}{c} a_{i-1}+1\dots M_i-1\; M_i+1\dots a_i-1 \\ a_{i-1}+1\dots M_i-1\; M_i+1\dots a_i-1 \end{array}}$, is simply the block $\boxed{\begin{array}{c} a_{i-1}+1\dots a_i-2 \\ a_{i-1}+1\dots a_i-2 \end{array}}$. In particular, if $D(u)=\{2\}$ then (u,v) is the following pair:

$$\begin{array}{l} u = 1\; n-1 \\ v = n\quad 2 \end{array}\boxed{\begin{array}{c} 2\quad 3\dots a_3-2 \\ a_3-1\; 3\dots a_3-2 \end{array}}\boxed{\begin{array}{c} a_3-1 a_3\dots a_4-2 \\ a_4-1 a_3\dots a_4-2 \end{array}}\boxed{\begin{array}{c} a_4-1 a_4\dots a_5-2 \\ a_5-1 a_4\dots a_5-2 \end{array}}\begin{array}{c} \dots\dots \\ \dots\dots \end{array}$$

$$\boxed{\begin{array}{c} a_{k-2}-1 a_{k-2}\dots a_{k-1}-2 \\ a_{k-1}-1 a_{k-2}\dots a_{k-1}-2 \end{array}}\begin{array}{c} a_{k-1}-1 \\ n-2 \end{array}\boxed{\begin{array}{c} a_{k-1}\dots n-3 \\ a_{k-1}\dots n-3 \end{array}}\begin{array}{c} n-2\; n \\ n-1\;1 \end{array}$$

Note that under the hypothesis that $D(u)=\{2\}$, we have that $a_2=3$, otherwise it will be $v(2)=u(a_2)=a_2-1>2=v(3)$, so that $2\in D(v)$, which is a contradiction. Otherwise if $a_2-1\in D(u)$, $a_2>3$.

We will prove, by induction on $|D(v)|$ that $(u,v)\sim(x,(1,n))$, where x is a pattern of type $k12\dots k-1$.
Therefore, our goal is to transform the pair (u,v) into a pair (u_1,v_1), which satisfies the conditions of the statement and such that $|D(v_1)|<|D(v)|$, by a sequence of operations which are allowed by Proposition 2.2.
We describe this sequence of operations, looking at the pair (u,v) as a matrix of dimension $2\times n$ and focusing our attention on the effect of each operation to its columns. We have emphasized the blocks of (u,v), whose transformation, is the central point to explain the whole transformation.

The first operation is the multiplication of (u, v) to the right by w_0, i.e. we reverse the permutations; in terms of columns we read the columns of (u, v) starting from the right to the left and swapping the two elements inside each column. The result is:

$$\begin{array}{rl|c|l}
\cline{3-3}
vw_0 & = 1\; n-1 & n-2 \dots M_k+1\; M_k-1 \dots a_{k-1} & \;M_k \\
uw_0 & = n \;\; M_k & n-2 \dots M_k+1\; M_k-1 \dots a_{k-1} & M_{k-1} \\
\cline{3-3}
\end{array}$$

$$\begin{array}{|c|l}
\hline
a_{k-1}-1 \dots M_{k-1}+1\; M_{k-1}-1 \dots a_{k-2} & M_{k-1} \\
a_{k-1}-1 \dots M_{k-1}+1\; M_{k-1}-1 \dots a_{k-2} & M_{k-2} \\
\hline
\end{array}$$

$$\begin{array}{|c|l}
\hline
a_{k-2}-1 \dots M_{k-2}+1\; M_{k-2}-1 \dots a_{k-3} & \dots\dots \\
a_{k-2}-1 \dots M_{k-2}+1\; M_{k-2}-1 \dots a_{k-3} & \dots\dots \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|}
\hline
a_5-1 \dots M_5+1\; M_5-1 \dots a_4 & M_5 & a_4-1 \dots M_4+1\; M_4-1 \dots a_3 \\
a_5-1 \dots M_5+1\; M_5-1 \dots a_4 & M_4 & a_4-1 \dots M_4+1\; M_4-1 \dots a_3 \\
\hline
\end{array}$$

$$\begin{array}{r|c|c|c|l}
\cline{2-2}\cline{4-4}
M_4 & a_3-1 \dots M_3+1\; M_3-1 \dots a_2 & M_3 & a_2-1 \dots 3 & \;\;2 \;\;\; n \\
M_3 & a_3-1 \dots M_3+1\; M_3-1 \dots a_2 & 2 & a_2-1 \dots 3 & n-1\; 1 \\
\cline{2-2}\cline{4-4}
\end{array}$$

We reorder the elements inside each boxed blocks, since they are in decreasing order; this corresponds to multiply both permutations by suitable descents. We indicate the result by the prefix "R":

$$\begin{array}{rl|c|l}
\cline{3-3}
Rvw_0 & = 1\; \mathbf{n-1} & a_{k-1} \dots M_k-1\; M_k+1 \dots n-2 & \;\mathbf{M_k} \\
Ruw_0 & = n \;\; \mathbf{M_k} & a_{k-1} \dots M_k-1\; M_k+1 \dots n-2 & \mathbf{M_{k-1}} \\
\cline{3-3}
\end{array}$$

$$\begin{array}{|c|l}
\hline
a_{k-2} \dots M_{k-1}-1\; M_{k-1}+1 \dots a_{k-1}-1 & \mathbf{M_{k-1}} \\
a_{k-2} \dots M_{k-1}-1\; M_{k-1}+1 \dots a_{k-1}-1 & \mathbf{M_{k-2}} \\
\hline
\end{array}$$

$$\begin{array}{|c|l}
\hline
a_{k-3} \dots M_{k-2}-1\; M_{k-2}+1 \dots a_{k-2}-1 & \mathbf{M_{k-2}} \\
a_{k-3} \dots M_{k-2}-1\; M_{k-2}+1 \dots a_{k-2}-1 & \mathbf{M_{k-3}} \\
\hline
\end{array}$$

$$\begin{array}{r|c|c|c|l}
\cline{2-2}\cline{4-4}
\dots\dots & a_4 \dots M_5-1\; M_5+1 \dots a_5-1 & \mathbf{M_5} & a_3 \dots M_4-1\; M_4+1 \dots a_4-1 & \mathbf{M_4} \\
\dots\dots & a_4 \dots M_5-1\; M_5+1 \dots a_5-1 & \mathbf{M_4} & a_3 \dots M_4-1\; M_4+1 \dots a_4-1 & \mathbf{M_3} \\
\cline{2-2}\cline{4-4}
\end{array}$$

$$\begin{array}{|c|c|c|l}
\cline{1-1}\cline{3-3}
a_2 \dots M_3-1\; M_3+1 \dots a_3-1 & \mathbf{M_3} & 3 \dots a_2-1 & \;\;2 \;\;\; n \\
a_2 \dots M_3-1\; M_3+1 \dots a_3-1 & \mathbf{2} & 3 \dots a_2-1 & n-1\; 1 \\
\cline{1-1}\cline{3-3}
\end{array}$$

Next we want to shift these blocks, to "eliminate" the remaining common descents, we will do this in different steps, to better explain this passage; we indicate the final result with the prefix "S".

We start the shifting by moving the columns in bold letters as follows: the ones of type $\boxed{\begin{array}{c} M_h \\ M_{h-1} \end{array}}$ between the columns $\begin{array}{cc} M_h-1 & M_h+1 \\ M_h-1 & M_h+1 \end{array}$, $\forall h \in [3,k]$; once we have done this, we move $\boxed{\begin{array}{c} n-1 \\ M_k \end{array}}$ between $\begin{array}{cc} M_k & M_k+1 \\ M_{k-1} & M_k+1 \end{array}$. We obtain the following pair:

$$\begin{array}{c|c|c|c|}
\cline{2-4}
1 & a_{k-1}\dots M_k-1\ \ M_k & n-1\ M_k+1\dots n-2 & a_{k-2}\dots M_{k-1}-1\ M_{k-1} \\
n & a_{k-1}\dots M_k-1\ M_{k-1} & M_k\ \ M_k+1\dots n-2 & a_{k-2}\dots M_{k-1}-1\ M_{k-2} \\
\cline{2-4}
\end{array}$$

$$\begin{array}{|c|c|c|c}
\hline
M_{k-1}+1\dots a_{k-1}-1 & a_{k-3}\dots M_{k-2}-1\ M_{k-2} & M_{k-2}+1\dots a_{k-2}-1 & \dots\dots \\
M_{k-1}+1\dots a_{k-1}-1 & a_{k-3}\dots M_{k-2}-1\ M_{k-3} & M_{k-2}+1\dots a_{k-2}-1 & \dots\dots \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|}
\hline
a_4\dots M_5-1\ M_5 & M_5+1\dots a_5-1 & a_3\dots M_4-1\ M_4 \\
a_4\dots M_5-1\ M_4 & M_5+1\dots a_5-1 & a_3\dots M_4-1\ M_3 \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|c|c}
\hline
M_4+1\dots a_4-1 & a_2\dots M_3-1\ M_3 & M_3+1\dots a_3-1 & 3\dots a_2-1 & 2\quad n \\
M_4+1\dots a_4-1 & a_2\dots M_3-1\ \ 2 & M_3+1\dots a_3-1 & 3\dots a_2-1 & n-1\ 1 \\
\hline
\end{array}$$

Now we move the second block to occupy the positions corresponding to the values of its second row; moreover we move the block $\boxed{\begin{array}{c} a_2\dots M_3-1\ \mathbf{M_3} \\ a_2\dots M_3-1\ \ \mathbf{2} \end{array}}$ to the positions $[2, M_3-a_2+1]$:

$$\begin{array}{c|c|c|c|}
\cline{2-4}
1 & a_2\dots M_3-1\ M_3 & a_{k-1}\dots M_k-1\ \ M_k & a_{k-2}\dots M_{k-1}-1\ M_{k-1} \\
n & a_2\dots M_3-1\ \ 2 & a_{k-1}\dots M_k-1\ M_{k-1} & a_{k-2}\dots M_{k-1}-1\ M_{k-2} \\
\cline{2-4}
\end{array}$$

$$\begin{array}{|c|c|c|c}
\hline
M_{k-1}+1\dots a_{k-1}-1 & a_{k-3}\dots M_{k-2}-1\ M_{k-2} & M_{k-2}+1\dots a_{k-2}-1 & \dots\dots \\
M_{k-1}+1\dots a_{k-1}-1 & a_{k-3}\dots M_{k-2}-1\ M_{k-3} & M_{k-2}+1\dots a_{k-2}-1 & \dots\dots \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|}
\hline
a_4\dots M_5-1\ M_5 & M_5+1\dots a_5-1 & a_3\dots M_4-1\ M_4 \\
a_4\dots M_5-1\ M_4 & M_5+1\dots a_5-1 & a_3\dots M_4-1\ M_3 \\
\hline
\end{array}$$

$$\begin{array}{|c|c|c|c|c}
\hline
M_4+1\dots a_4-1 & M_3+1\dots a_3-1 & 3\dots a_2-1 & n-1\ M_k+1\dots n-2 & 2\quad n \\
M_4+1\dots a_4-1 & M_3+1\dots a_3-1 & 3\dots a_2-1 & M_k\ \ M_k+1\dots n-2 & n-1\ 1 \\
\hline
\end{array}$$

At this point one has different chances to continue the shifting of the blocks:

- if we move to the left the blocks, starting from the last but one we have to lock at the upper element of the columns $\boxed{\begin{array}{c} M_h \\ M_{h-1} \end{array}}$;

- if we move to the right the blocks, starting from the third one we have to concentrate on the lower elements of $\boxed{\begin{array}{c} M_h \\ M_{h-1} \end{array}}$.

We choose the second way; in each case the column $\boxed{\begin{array}{c} M_h \\ M_{h-1} \end{array}}$, is the leading element for its block, in the following sense:
we move the block $\boxed{\begin{array}{l} a_{k-1} \dots M_k - 1 \;\; M_k \\ a_{k-1} \dots M_k - 1 \, M_{k-1} \end{array}}$, just before the block $\boxed{\begin{array}{l} M_{k-1} + 1 \dots a_{k-1} - 1 \\ M_{k-1} + 1 \dots a_{k-1} - 1 \end{array}}$.
Now we have a new block: $\boxed{\begin{array}{l} a_{k-1} \dots M_k - 1 \;\; M_k \;\; M_{k-1} + 1 \dots a_{k-1} - 1 \\ a_{k-1} \dots M_k - 1 \, M_{k-1} \, M_{k-1} + 1 \dots a_{k-1} - 1 \end{array}}$ that can be moved to the right just before the column $\begin{array}{c} n-1 \\ M_k \end{array}$. We write the new pair of permutations:

$$\begin{array}{c} 1 \\ n \end{array} \boxed{\begin{array}{l} a_2 \dots M_3 - 1 \, M_3 \\ a_2 \dots M_3 - 1 \;\; 2 \end{array}} \boxed{\begin{array}{l} a_{k-2} \dots M_{k-1} - 1 \, M_{k-1} \\ a_{k-2} \dots M_{k-1} - 1 \, M_{k-2} \end{array}} \boxed{\begin{array}{l} a_{k-3} \dots M_{k-2} - 1 \, M_{k-2} \\ a_{k-3} \dots M_{k-2} - 1 \, M_{k-3} \end{array}}$$

$$\boxed{\begin{array}{l} M_{k-2} + 1 \dots a_{k-2} - 1 \\ M_{k-2} + 1 \dots a_{k-2} - 1 \end{array}} \begin{array}{c} \dots\dots \\ \dots\dots \end{array} \boxed{\begin{array}{l} a_4 \dots M_5 - 1 \, M_5 \\ a_4 \dots M_5 - 1 \, M_4 \end{array}} \boxed{\begin{array}{l} M_5 + 1 \dots a_5 - 1 \\ M_5 + 1 \dots a_5 - 1 \end{array}} \boxed{\begin{array}{l} a_3 \dots M_4 - 1 \, M_4 \\ a_3 \dots M_4 - 1 \, M_3 \end{array}}$$

$$\boxed{\begin{array}{l} M_4 + 1 \dots a_4 - 1 \\ M_4 + 1 \dots a_4 - 1 \end{array}} \boxed{\begin{array}{l} M_3 + 1 \dots a_3 - 1 \\ M_3 + 1 \dots a_3 - 1 \end{array}} \boxed{\begin{array}{l} 3 \dots a_2 - 1 \\ 3 \dots a_2 - 1 \end{array}}$$

$$\boxed{\begin{array}{l} a_{k-1} \dots M_k - 1 \;\; M_k \;\; M_{k-1} + 1 \dots a_{k-1} - 1 \\ a_{k-1} \dots M_k - 1 \, M_{k-1} \, M_{k-1} + 1 \dots a_{k-1} - 1 \end{array}} \boxed{\begin{array}{l} n - 1 \, M_k + 1 \dots n - 2 \\ \;\; M_k \;\; M_k + 1 \dots n - 2 \end{array}} \begin{array}{cc} 2 & n \\ n-1 & 1 \end{array}$$

Now we do the same with $\boxed{\begin{array}{l} a_{k-2} \dots M_{k-1} - 1 \, M_{k-1} \\ a_{k-2} \dots M_{k-1} - 1 \, M_{k-2} \end{array}}$ and we obtain:

$$\begin{array}{c} 1 \\ n \end{array} \boxed{\begin{array}{l} a_2 \dots M_3 - 1 \, M_3 \\ a_2 \dots M_3 - 1 \;\; 2 \end{array}} \boxed{\begin{array}{l} a_{k-3} \dots M_{k-2} - 1 \, M_{k-2} \\ a_{k-3} \dots M_{k-2} - 1 \, M_{k-3} \end{array}} \begin{array}{c} \dots\dots \\ \dots\dots \end{array} \boxed{\begin{array}{l} a_4 \dots M_5 - 1 \, M_5 \\ a_4 \dots M_5 - 1 \, M_4 \end{array}}$$

$$\boxed{\begin{array}{l} M_5 + 1 \dots a_5 - 1 \\ M_5 + 1 \dots a_5 - 1 \end{array}} \boxed{\begin{array}{l} a_3 \dots M_4 - 1 \, M_4 \\ a_3 \dots M_4 - 1 \, M_3 \end{array}} \boxed{\begin{array}{l} M_4 + 1 \dots a_4 - 1 \\ M_4 + 1 \dots a_4 - 1 \end{array}} \boxed{\begin{array}{l} M_3 + 1 \dots a_3 - 1 \\ M_3 + 1 \dots a_3 - 1 \end{array}}$$

$$\boxed{\begin{array}{l} 3 \dots a_2 - 1 \\ 3 \dots a_2 - 1 \end{array}} \boxed{\begin{array}{l} a_{k-2} \dots M_{k-1} - 1 \, M_{k-1} \, M_{k-2} + 1 \dots a_{k-2} - 1 \\ a_{k-2} \dots M_{k-1} - 1 \, M_{k-2} \, M_{k-2} + 1 \dots a_{k-2} - 1 \end{array}}$$

$$\boxed{\begin{array}{l} a_{k-1} \dots M_k - 1 \;\; M_k \;\; M_{k-1} + 1 \dots a_{k-1} - 1 \\ a_{k-1} \dots M_k - 1 \, M_{k-1} \, M_{k-1} + 1 \dots a_{k-1} - 1 \end{array}} \boxed{\begin{array}{l} n - 1 \, M_k + 1 \dots n - 2 \\ \;\; M_k \;\; M_k + 1 \dots n - 2 \end{array}} \begin{array}{cc} 2 & n \\ n-1 & 1 \end{array}$$

We go on in this way and clearly the final result is:

$$
\begin{array}{ll}
SRvw_0 = 1 & \boxed{\begin{array}{l} a_2 \ldots M_3 - 1\ M_3\ 3 \ldots a_2 - 1 \\ a_2 \ldots M_3 - 1\ \ 2\ \ 3 \ldots a_2 - 1 \end{array}} \boxed{\begin{array}{l} a_3 \ldots M_4 - 1\ M_4\ M_3 + 1 \ldots a_3 - 1 \\ a_3 \ldots M_4 - 1\ M_3\ M_3 + 1 \ldots a_3 - 1 \end{array}} \\
SRuw_0 = n &
\end{array}
$$

$$
\boxed{\begin{array}{l} a_4 \ldots M_5 - 1\ M_5\ M_4 + 1 \ldots a_4 - 1 \\ a_4 \ldots M_5 - 1\ M_4\ M_4 + 1 \ldots a_4 - 1 \end{array}} \boxed{\begin{array}{l} a_5 \ldots M_6 - 1\ M_6\ M_5 + 1 \ldots a_5 - 1 \\ a_5 \ldots M_6 - 1\ M_5\ M_5 + 1 \ldots a_5 - 1 \end{array}} \begin{array}{l} \ldots\ldots \\ \ldots\ldots \end{array}
$$

$$
\boxed{\begin{array}{l} a_{k-2} \ldots M_{k-1} - 1\ M_{k-1}\ M_{k-2} + 1 \ldots a_{k-2} - 1 \\ a_{k-2} \ldots M_{k-1} - 1\ M_{k-2}\ M_{k-2} + 1 \ldots a_{k-2} - 1 \end{array}}
$$

$$
\boxed{\begin{array}{l} a_{k-1} \ldots M_k - 1\ \ M_k\ \ M_{k-1} + 1 \ldots a_{k-1} - 1 \\ a_{k-1} \ldots M_k - 1\ M_{k-1}\ M_{k-1} + 1 \ldots a_{k-1} - 1 \end{array}}
$$

$$
\begin{array}{l} n - 1\ M_k + 1 \ldots n - 2\quad 2\quad n \\ \ \ M_k\ \ M_k + 1 \ldots n - 2\ n - 1\ 1 \end{array}
$$

Now we calculate the inverses:

$$
\begin{array}{ll}
(SRvw_0)^{-1} = 1 \qquad n - 1 \qquad M_3 - a_2 + 3 \ldots M_3 - 1 & \boxed{\begin{array}{l} 2 \ldots M_3 - a_2 + 1 \\ 2 \ldots M_3 - a_2 + 1 \end{array}} \\
(SRuw_0)^{-1} = n\ M_3 - a_2 + 2\ M_3 - a_2 + 3 \ldots M_3 - 1 &
\end{array}
$$

$$
\begin{array}{r} M_3 - a_2 + 2\ \ M_3 + M_4 - a_3 + 1 \ldots M_4 - 1 \\ M_3 + M_4 - a_3\ M_3 + M_4 - a_3 + 1 \ldots M_4 - 1 \end{array} \boxed{\begin{array}{l} M_3 \ldots M_3 + M_4 - a_3 - 1 \\ M_3 \ldots M_3 + M_4 - a_3 - 1 \end{array}}
$$

$$
\begin{array}{r} M_3 + M_4 - a_3\ M_4 + M_5 - a_4 + 1 \ldots M_5 - 1 \\ M_4 + M_5 - a_4\ M_4 + M_5 - a_4 + 1 \ldots M_5 - 1 \end{array} \boxed{\begin{array}{l} M_4 \ldots M_4 + M_5 - a_4 - 1 \\ M_4 \ldots M_4 + M_5 - a_4 - 1 \end{array}}
$$

$$
\begin{array}{r} M_4 + M_5 - a_4\ M_5 + M_6 - a_5 + 1 \ldots M_6 - 1 \\ M_5 + M_6 - a_5\ M_5 + M_6 - a_5 + 1 \ldots M_6 - 1 \end{array} \boxed{\begin{array}{l} M_5 \ldots M_5 + M_6 - a_5 - 1 \\ M_5 \ldots M_5 + M_6 - a_5 - 1 \end{array}}
$$

$$
\begin{array}{l} \ldots\ldots \\ \ldots\ldots \end{array} \boxed{\begin{array}{l} M_{k-2} \ldots M_{k-2} + M_{k-1} - a_{k-2} - 1 \\ M_{k-2} \ldots M_{k-2} + M_{k-1} - a_{k-2} - 1 \end{array}} \begin{array}{l} M_{k-2} + M_{k-1} - a_{k-2} \\ \ \ M_k + M_{k-1} - a_{k-1} \end{array}
$$

$$
\begin{array}{r} M_k + M_{k-1} - a_{k-1} + 1 \ldots M_k - 1 \\ M_k + M_{k-1} - a_{k-1} + 1 \ldots M_k - 1 \end{array} \boxed{\begin{array}{l} M_{k-1} \ldots M_k + M_{k-1} - a_{k-1}\ M_k + M_{k-1} - a_{k-1} - 1 \\ M_{k-1} \ldots M_k + M_{k-1} - a_{k-1}\ M_k + M_{k-1} - a_{k-1} - 1 \end{array}}
$$

$$
\begin{array}{l} M_k + M_{k-1} - a_{k-1}\ M_k + 1 \ldots n - 2\ \ M_k\ \ n \\ \qquad\qquad M_k \qquad\qquad M_k + 1 \ldots n - 2\ n - 1\ 1 \end{array}
$$

Note that, since by definition $M_i < a_i$ and then $M_i + M_{i+1} - a_i < M_{i+1}$, we can shift each boxed blocks properly to the left to obtain a pair of permutations with no common left and right descents:

$$
\begin{array}{ll}
u_1 = 1 \\
v_1 = n
\end{array}
\boxed{\begin{array}{l}
2 \dots M_3 - a_2 + 1 \\
2 \dots M_3 - a_2 + 1
\end{array}}
\begin{array}{c}
n - 1 \\
M_3 - a_2 + 2
\end{array}
\boxed{\begin{array}{l}
M_3 - a_2 + 3 \dots M_3 - 1 \\
M_3 - a_2 + 3 \dots M_3 - 1
\end{array}}
\begin{array}{l}
M_3 - a_2 + 2 \\
M_3 + M_4 - a_3
\end{array}
$$

$$
\boxed{\begin{array}{l}
M_3 \dots M_3 + M_4 - a_3 - 1 \; M_3 + M_4 - a_3 + 1 \dots M_4 - 1 \\
M_3 \dots M_3 + M_4 - a_3 - 1 \; M_3 + M_4 - a_3 + 1 \dots M_4 - 1
\end{array}}
\begin{array}{l}
M_3 + M_4 - a_3 \\
M_4 + M_5 - a_4
\end{array}
$$

$$
\boxed{\begin{array}{l}
M_4 \dots M_4 + M_5 - a_4 - 1 \; M_4 + M_5 - a_4 + 1 \dots M_5 - 1 \\
M_4 \dots M_4 + M_5 - a_4 - 1 \; M_4 + M_5 - a_4 + 1 \dots M_5 - 1
\end{array}}
\begin{array}{l}
M_4 + M_5 - a_4 \\
M_5 + M_6 - a_5
\end{array}
$$

$$
\begin{array}{l}
\dots\dots \\
\dots\dots
\end{array}
\boxed{\begin{array}{l}
M_{k-2} \dots M_{k-2} + M_{k-1} - a_{k-2} - 1 \; M_{k-2} + M_{k-1} - a_{k-2} + 1 \\
M_{k-2} \dots M_{k-2} + M_{k-1} - a_{k-2} - 1 \; M_{k-2} + M_{k-1} - a_{k-2} + 1
\end{array}}
$$

$$
\begin{array}{r}
M_{k-2} + M_{k-1} - a_{k-2} \\
M_k + M_{k-1} - a_{k-1}
\end{array}
\boxed{\begin{array}{l}
M_{k-1} \dots M_k + M_{k-1} - a_{k-1} - 1 \; M_k + M_{k-1} - a_{k-1} + 1 \\
M_{k-1} \dots M_k + M_{k-1} - a_{k-1} - 1 \; M_k + M_{k-1} - a_{k-1} + 1
\end{array}}
$$

$$
\begin{array}{lllll}
\dots M_k - 1 & M_k + M_{k-1} - a_{k-1} & M_k + 1 \dots n - 2 & M_k & n \\
\dots M_k - 1 & M_k & M_k + 1 \dots n - 2 & n - 1 & 1
\end{array}
$$

Summarizing we have done the following sequence of operations:

$$
\begin{aligned}
(u, v) \longrightarrow (vw_0, uw_0) \longrightarrow (Rvw_0, Ruw_0) \longrightarrow (SRvw_0, SRuw_0) \\
\longrightarrow ((SRvw_0)^{-1}, (SRuw_0)^{-1}) \longrightarrow (u_1, v_1)
\end{aligned}
$$

We define the function

$$
f : (u, v) \mapsto (u_1, v_1) \tag{3.1}
$$

as the composition of the previous operations, clearly f acts inside the equivalence class of (u, v) by Proposition 2.2

Moreover $f(u, v) = (u_1, v_1)$ is a pair which satisfies the initial hypothesis of the theorem. The action on the descents is the following: $|D(u)| = |D(u_1)|$, since $|\{M_3 - a_2 + 2\} \cup \{M_i - 1 : i \in [3, k]\}|$, while $D(v_1) = \{1, n-1\} \cup \{M_i : i \in [3, k-1]\}$ and then $|D(v_1)| = |D(v)| - 1$.

Therefore by induction $(u_1, v_1) \sim (x, (1, n))$ and this concludes the proof. $\square$

Remark on the function f

The proof of the Theorem 3.3 is based on the induction on $|D(v)|$, and then on the function f. We can observe that if we take (u, v) under the conditions of Theorem 3.3 then $f^{k-2}(u, v)$ is a pair of permutations composed by a k-cycle and $(1, n)$ when

$D(v)$ is the maximum possible and $D(u) = \{2\}$. This is because of the action of f on the descents of the greatest permutation.

Consider to implement in a computer a program that applies the f, $k-2$-times to reduce the initial pair (u, v) to a pair in which the second permutation is $(1, n)$. We have observed that if we take the initial pair (u, v) such that $D(v)$ is properly contained in $\{1, a_2, a_3, \ldots, a_{k-1}, n-1\}$ then since the function f reduces the number of descents at each step, we can say that there exists an index s such that $1 \leq s \leq k-2$, such that $f^s(u, v) = (x, (1, n))$, where x is $k-cycle$. This number s depends on the positions of the descents in $D(v)$, but so far we don't know how; we will explain in the next example what we mean.
Consider the following pair of permutations in S_{15}:

$$\begin{aligned} u &= 1\ 14\ 3\ 4\ 2\ 5\ 7\ 6\ 9\ 10\ 8\ 11\ 13\ 12\ 15 \\ v &= 15\ 2\ 3\ 4\ 6\ 5\ 7\ 8\ 9\ 10\ 12\ 11\ 13\ 14\ 1 \end{aligned}$$

We observe that u contains a pattern 51234 at the positions $2, 5, 8, 11, 14$, $D(u) = \{2, 4, 7, 10, 13\}$ and $v = u(1, 15)(2, 5, 8, 11, 13)$, $D(v) = \{1, 5, 12, 14\}$. So this pair satisfies the conditions of Theorem 3.3 and v is a permutation which does not contain the descent $\{8\}$. We show that $f(u, v) = (u_1, (1, n))$, i.e that $s = 1$, for this case:

$$\begin{aligned} vw_0 &= 1\ 14 \boxed{13\ 11}\ 12 \boxed{10\ 9}\ 8 \boxed{7\ 5}\ 6 \boxed{4\ 3}\ 2\ 15 \\ uw_0 &= 15\ 12 \boxed{13\ 11}\ 8 \boxed{10\ 9}\ 6 \boxed{7\ 5}\ 2 \boxed{4\ 3}\ 14\ 1 \end{aligned}$$

$$\begin{aligned} Rvw_0 &= 1\ 14 \boxed{11\ 13}\ 12 \boxed{9\ 10}\ 8\ 5\ 7\ 6 \boxed{3\ 4}\ 2\ 15 \\ Ruw_0 &= 15\ 12 \boxed{11\ 13}\ 8 \boxed{9\ 10}\ 6\ 5\ 7\ 2 \boxed{3\ 4}\ 14\ 1 \end{aligned}$$

$$\begin{aligned} SRvw_0 &= 1\ 5\ 6\ 3\ 4\ 8\ 7\ 12\ 9\ 10\ 11\ 14\ 13\ 2\ 15 \\ SRuw_0 &= 15\ 5\ 2\ 3\ 4\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 1 \end{aligned}$$

$$\begin{aligned} (SRvw_0)^{-1} &= 1\ 14\ 4\ 5 \boxed{2}\ 3\ 7\ 6\ 9\ 10\ 11\ 8\ 13\ 12\ 15 \\ (SRuw_0)^{-1} &= 15\ 3\ 4\ 5 \boxed{2}\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 1 \end{aligned}$$

$$\begin{aligned} u_1 &= 1\ 2\ 14\ 4\ 5\ 3\ 7\ 6\ 9\ 10\ 11\ 8\ 13\ 12\ 15 \\ v_1 &= 15\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 1 \end{aligned}$$

(We have also seen that if $\{D(v)\} = \{8\}$, then the number $s = k-2 = 3$ as the case in which we have all the possible descents for v, but we avoid to show this.)

We also observe that we are able to write (u, v), in every case of $D(v)$, with the representation given in the Theorem 3.3, i.e. using the parameters A_i, in fact if

- $a_i \notin D(v)$, and $a_i - 1 \notin D(u)$ it forces that $a_{i+1} = a_i + 1$, while if $a_i - 1 \in D(u)$ and $a_i \notin D(v)$, then $A_i = \emptyset$ and $u(a_i) < u(a_i + 1)$

- $\forall m \in [2, k-1]$, $a_m \notin D(v)$ if and only if $a_m = a_{m+1} - 1$:
  suppose that for a fixed $m \in [2, k-1]$, $a_m \notin D(v)$, then $v(a_m) = u(a_{m+1}) = a_{m+1} - 1 < v(a_m + 1) = u(a_m + 1) = a_m$ which is a contradiction unless $a_m = a_{m+1} - 1$. On the other hand, if for a fixed m, $a_m = a_{m+1}$, then $a_m + 1 = a_{m+1}$, thus $v(a_m) = a_{m+1} - 1 < v(a_{m+1}) = a_{m+2} - 1$, which means that $a_m \notin D(v)$.

Nevertheless if we apply f, $k-2$-times on (u, v) the result does not change, as we prove in Lemma 3.4.

Lemma 3.4. *The function f, defined in 3.1 of the previous proof, acts as the identity on the pairs $(u, (1, n))$ such that $u = (a_k, a_{k-1}, a_{k-2}, \ldots, a_3, a_2, a_1)$, i.e. $f(u, (1, n)) = (u, (1, n))$.*

Proof. From the hypotheses follows that

$$\begin{aligned} u &= 1\,2 \ldots a_1 - 1\; a_k\; a_1 + 1 \ldots a_2 - 1\; a_1\; a_2 + 1 \ldots a_3 - 1\; a_2\; a_3 + 1 \ldots a_4 - 1 \ldots\ldots \\ v &= n\,2 \ldots a_1 - 1\; a_1\; a_1 + 1 \ldots a_2 - 1\; a_2\; a_2 + 1 \ldots a_3 - 1\; a_3\; a_3 + 1 \ldots a_4 - 1 \ldots\ldots \end{aligned}$$

$$\begin{aligned} &a_{k-2} - 1\; a_{k-3}\; a_{k-2} + 1 \ldots a_{k-1} - 1\; a_{k-2}\; a_{k-1} + 1 \ldots a_k - 1\; a_{k-1}\; a_k + 1 \ldots n - 1\; n \\ &a_{k-2} - 1\; a_{k-2}\; a_{k-2} + 1 \ldots a_{k-1} - 1\; a_{k-1}\; a_{k-1} + 1 \ldots a_k - 1\;\; a_k\;\; a_k + 1 \ldots n - 1\; 1 \end{aligned}$$

We calculate $f(u, v)$, recalling that this means the following composition:

$$\begin{aligned} (u, v) \longrightarrow (vw_0, uw_0) \longrightarrow (Rvw_0, Ruw_0) \longrightarrow (SRvw_0, SRuw_0) \\ \longrightarrow ((SRvw_0)^{-1}, (SRuw_0)^{-1}) \end{aligned}$$

$$\begin{array}{rl|c|c|c|} \cline{3-5} vw_0 & = 1 & n - 1 \ldots a_k + 1\;\; a_k & a_k - 1 \ldots a_{k-1} + 1\; a_{k-1} & a_{k-1} - 1 \ldots a_{k-2} + 1\; a_{k-2} \\ uw_0 & = n & n - 1 \ldots a_k + 1\; a_{k-1} & a_k - 1 \ldots a_{k-1} + 1\; a_{k-2} & a_{k-1} - 1 \ldots a_{k-2} + 1\; a_{k-3} \\ \cline{3-5} \end{array}$$

$$\begin{array}{l|c|c|c|c|l} \cline{2-5} a_{k-2} - 1 \ldots\ldots & a_4 - 1 \ldots a_3 + 1\; a_3 & a_3 - 1 \ldots a_2 + 1\; a_2 & a_2 - 1 \ldots a_1 + 1 & a_1\; a_1 - 1 \ldots 2 & n \\ a_{k-2} - 1 \ldots\ldots & a_4 - 1 \ldots a_3 + 1\; a_2 & a_3 - 1 \ldots a_2 + 1\; a_1 & a_2 - 1 \ldots a_1 + 1 & a_k\; a_1 - 1 \ldots 2 & 1 \\ \cline{2-5} \end{array}$$

Now we put the values inside the boxed blocks in their increasing order:

$$\begin{array}{rl|c|c|c|} \cline{3-5} Rvw_0 & = 1 & a_k\;\; a_k + 1 \ldots n - 1 & a_{k-1}\; a_{k-1} + 1 \ldots a_k - 1 & a_{k-2}\; a_{k-2} + 1 \ldots a_{k-1} - 1 \\ Ruw_0 & = n & a_{k-1}\; a_k + 1 \ldots n - 1 & a_{k-2}\; a_{k-1} + 1 \ldots a_k - 1 & a_{k-3}\; a_{k-2} + 1 \ldots a_{k-1} - 1 \\ \cline{3-5} \end{array}$$

$$
\begin{array}{l|l|l|l|l|l}
\ldots\ldots & a_3\, a_3+1\ldots a_4-1 & a_2\, a_2+1\ldots a_3-1 & a_1+1\ldots a_2-1 & 2\ldots a_1-1\, a_1 & n \\
\ldots\ldots & a_2\, a_3+1\ldots a_4-1 & a_1\, a_2+1\ldots a_3-1 & a_1+1\ldots a_2-1 & 2\ldots a_1-1\, a_k & 1
\end{array}
$$

We shift the blocks, in the usual way:

$$
\begin{aligned}
SRvw_0 &= 1\, 2\ldots a_1-1\, a_2\, a_1+1\ldots a_2-1\, a_3\, a_2+1\ldots a_3-1\, a_4\, a_3+1\ldots a_4-1 \\
SRuw_0 &= n\, 2\ldots a_1-1\, a_1\, a_1+1\ldots a_2-1\, a_2\, a_2+1\ldots a_3-1\, a_3\, a_3+1\ldots a_4-1
\end{aligned}
$$

$$
\begin{array}{l}
\ldots\ldots a_{k-2}-1\, a_{k-1}\, a_{k-2}+1\ldots a_{k-1}-1\quad a_k\quad a_{k-1}+1\ldots a_k-1\, a_1\, a_k+1\ldots n-1\, n \\
\ldots\ldots a_{k-2}-1\, a_{k-2}\, a_{k-2}+1\ldots a_{k-1}-1\, a_{k-1}\, a_{k-1}+1\ldots a_k-1\, a_k\, a_k+1\ldots n-1\, 1
\end{array}
$$

Now it is $(SRvw_0)^{-1} = u$ since $SRvw_0 = (a_1, a_2, \ldots, a_{k-1,a_k})$ and $SRuw_0 = (1, n)$, as we wanted prove.

$\square$

Corollary 3.5. *Let $u \in S_n$ and $1 \le i < a_1 < a_2 < a_3 < \cdots < a_k < j \le n$, such that u contains a pattern $23\ldots(k-1)k1$ at the positions a_1, a_2, $\ldots$, a_k.*
If $u < v = u(i,j)(a_1, a_k, a_{k-1}, \ldots, a_2)$ and $D(u) \cap D(v) = \emptyset = D_L(u) \cap D_L(v)$ then

$$(u, v) \sim (x, (i, j))$$

where x is a k-cycle which is a $23\ldots(k-1)k1$ pattern.

Proof. From Lemma 3.2 we have that $u(i) < u(a_2)$ and that $u(a_1) < u(j)$.
We can assume, without loss of generality, that $i = 1$, $j = n$ and $u(1) = 1$, $u(n) = n$.
Moreover the hypothesis $D(u) \cap D(v) = \emptyset$ forces also that the values in the intervals $[a_m + 1, a_{m+1} - 1]$ are in increasing order, $\forall m \in [2, k-1]$.
The proof of this result can be done as the one of Theorem 3.3, but we simply use the symmetry between this family of permutations and the one of Theorem 3.3.
A pair (u, v) under these hypotheses can be written as follows:

$$
\begin{aligned}
u &= 1\, u(2)\ldots u(a_1-1)u(a_1)u(a_1+1)\ldots u(a_2-1)u(a_2)u(a_2+1)\ldots\ldots \\
v &= n\, u(2)\ldots u(a_1-1)u(a_k)u(a_1+1)\ldots u(a_2-1)u(a_1)u(a_2+1)\ldots\ldots
\end{aligned}
$$

$$
\begin{array}{l}
u(a_{k-1}-1)u(a_{k-1})u(a_{k-1}+1)\ \ldots u(a_k-1)u(a_k)u(a_k+1)\ldots u(n-1)\ \ n \\
u(a_{k-1}-1)u(a_{k-2})u(a_{k-1}+1)\ldots u(a_k-1)u(a_{k-1})u(a_k+1)\ldots u(n-1)\ 1
\end{array}
$$

From the hypotheses follows that $u(a_k) < u(a_1) < u(a_2) < u(a_3) < \cdots < u(a_{k-2}) < u(a_{k-1})$, then we can determine the values in which u^{-1} differ from v^{-1}, and we write these inverses:

$$\begin{aligned} u^{-1} &= 1 \dots a_k \dots a_1 \dots a_2 \dots a_3 \dots \dots a_{k-3} \dots a_{k-2} \dots a_{k-1} \dots n \\ v^{-1} &= n \dots a_1 \dots a_2 \dots a_3 \dots a_4 \dots \dots a_{k-2} \dots a_{k-1} \dots \ \ a_k \ \ \dots 1 \end{aligned}$$

Here we have omitted the positions in which the two permutations are coincident. We observe that u^{-1} contains a pattern $k12\dots k{-}1$ at the positions $u(a_k), u(a_1), u(a_2), u(a_3), \dots, u(a_{k-2}), u(a_{k-1})$, and $v^{-1} = u^{-1}(1,n)(u(a_k), u(a_1), \dots, u(a_{k-2}), u(a_{k-1}))$. Trivially $D(u) \cap D(v) = \emptyset = D_L(u) \cap D_L(v)$ is equivalent to the fact that $D(u^{-1}) \cap D(v^{-1}) = \emptyset = D_L(u^{-1}) \cap D_L(v^{-1})$, by Proposition 1.7. So we can apply Theorem 3.3 and the thesis follows. (Actually we obtain that $(u,v) \sim (x,(1,n))$ where x is a pattern of type $k12\dots$, but the pair $(x^{-1},(1,n))$ is as required).

□

Lemma 3.6. *Let* $1 < b_1 < b_2 < b_3 < \dots < b_k < n$.

$$\tilde{R}_{(b_k,b_{k-1},b_{k-2},\dots,b_3,b_2,b_1),(1,n)}(t) = (1+t^2)\tilde{R}_{(n+1-b_1,n+1-b_k,n+1-b_{k-1},\dots,n+1-b_3,n+1-b_2),(1,n-1)}(t)$$

Proof. We define $u = (b_k, b_{k-1}, b_{k-2}, \dots, b_3, b_2, b_1)$, and $v = (1,n)$.

We apply Theorem 1.17 to the descent $s_1 = (1,2) \in D(v)$ and then

$$\tilde{R}_{u,v}(t) = \tilde{R}_{us_1,vs_1}(t) + t\tilde{R}_{u,vs_1}(t)$$

We compute $\tilde{R}_{us_1,vs_1}(t) = \tilde{R}_{(us_1)^{-1},(vs_1)^{-1}}(t)$.

We observe that $us_1 = (2,1)(b_k, b_{k-1}, b_{k-2}, \dots, b_3, b_2, b_1)$ and $vs_1 = (n,1,2)$. Therefore $(us_1)^{-1} = (2,1)(b_k, b_1, b_2, b_3, \dots, b_{k-2}, b_{k-1})$, while $(vs_1)^{-1} = (n,2,1)$.

We multiply these permutations to the left by w_0 and we obtain:

$$\begin{matrix} w_0(vs_1)^{-1} \\ w_0(us_1)^{-1} \end{matrix} = \begin{matrix} 1 & \boxed{n} & n-2 \dots n+1-b_1 \dots n+1-b_2 \dots \dots n+1-b_{k-2} \dots \\ n-1 & \boxed{n} & n-2 \dots n+1-b_2 \dots n+1-b_4 \dots \dots n+1-b_{k-1} \dots \end{matrix}$$

$$\begin{matrix} n+1-b_{k-1} \dots n+1-b_k \dots 2\, n-1 \\ n+1-b_k \ \ \dots n+1-b_1 \dots 2 \quad 1 \end{matrix}$$

We reverse and we move $\begin{matrix} n \\ n \end{matrix}$ to the right, the result is the next pair:

$$\begin{matrix} 1 \quad 2 \dots n+1-b_1 \dots \ \ n+1-b_k \ \ \dots \dots n+1-b_{k-1} \dots \\ n-1\, 2 \dots n+1-b_k \dots n+1-b_{k-1} \dots \dots n+1-b_{k-2} \dots \end{matrix}$$

$$\begin{matrix} n+1-b_4 \dots n+1-b_3 \dots n+1-b_2 \dots n-2\, n-1 & \boxed{n} \\ n+1-b_3 \dots n+1-b_2 \dots n+1-b_1 \dots n-2 \quad 1 & \boxed{n} \end{matrix}$$

It remains to calculate $\tilde{R}_{u^{-1},(vs_1)^{-1}s_1}(t)$, since by Proposition 1.18, it is

$$\tilde{R}_{u,vs_1}(t) = \tilde{R}_{u^{-1},(vs_1)^{-1}}(t) = t\tilde{R}_{u^{-1},(vs_1)^{-1}s_1}(t)$$

In fact $s_1 \in D((vs_1)^{-1})$, but $s_1 \notin D(u^{-1})$ then by Theorem 1.17

$$\tilde{R}_{u^{-1},(vs_1)^{-1}s_1}(t) = \tilde{R}_{u^{-1}s_1,(vs_1)^{-1}s_1}(t) + t\tilde{R}_{u^{-1},(vs_1)^{-1}s_1}(t).$$

The first summand of the right hand side of the last equality vanishes since by Theorem 1.9

$$u^{-1}s_1 = (2,1)(b_k, b_1, b_2, \ldots, b_{k-2}, b_{k-1}) \nleq (vs_1)^{-1}s_1 = (2,n)$$

By Proposition 1.18, we have $\tilde{R}_{u^{-1},(2,n)}(t) = \tilde{R}_{w_0u^{-1}w_0,w_0(2,n)w_0}(t)$. We have that $(2,n)w_0 = 2n-1\ldots3n1$ then $w_0(2,n)w_0 = n-123\ldots n-21n$, while $u^{-1}w_0 = nn-1\ldots b_1\ldots b_k\ldots b_{k-1}\ldots\ldots b_4\ldots b_3\ldots b_2\ldots 21$ implies that $w_0u^{-1}w_0 = (n+1-b_1, n+1-b_k, n+1-b_{k-1}, \ldots, n+1-b_3, n+1-b_2)$.

Thus $\tilde{R}_{u,v}(t) = (1+t^2)\tilde{R}_{(n+1-b_1,n+1-b_k,n+1-b_{k-1},\ldots,n+1-b_3,n+1-b_2),(1,n-1)}(t)$.

□

We can exhibit the explicit formula:

Corollary 3.7. *Let $u \in S_n$ and $1 \leq i < a_1 < a_2 < a_3 < \cdots < a_k < j \leq n$, be such that:*

1. *u contains a pattern $k123\ldots k-1$ at the positions $a_1, a_2, \ldots, a_k$, and $u < v = u(i,j)(a_1, a_2, a_3, \ldots, a_k)$*
2. *u contains a pattern $23\ldots(k-1)k1$ at the positions $a_1, a_2, \ldots, a_k$ and $u < v = u(i,j)(a_1, a_k, a_{k-1}, \ldots, a_2)$*

Then

$$\tilde{R}_{u,v}(t) = t^{k+2}(1+t^2)^{(inv(v)-inv(u)-(k+2))/2}$$

Proof. It is enough to prove the statement for (i) for the equivalence proved in Corollary 3.5. By Theorem 1.17 and by Definition 2.2 we know that $(u,v) \sim (x,y)$ where $D(x) \cap D(y) = \emptyset = D_L(x) \cap D_L(y)$.

So we can assume that (u,v) satisfies the condition of Theorem 3.3 and, also by Lemma 3.2 that $i=1$, $j=n$ and $u(1)=1$, $u(n)=n$. This implies that $\tilde{R}_{u,v}(t) = \tilde{R}_{w,(1,n)}(t)$, where w is a k-cycle of type $(b_k, b_{k-1}, b_{k-2}, \ldots, b_3, b_2, b_1)$. We propose two ways of proving this result, the first one is the generalization of the method used in [17], Section 4, and consists on the induction on the distance between u and v. The second one is the application of Theorem 1.22.

First proof

We prove the corollary using the induction on $d(u,v) = max\{i \in [n] : u^{-1}(i) \neq$

$v^{-1}(i)\}$, i.e. the distance between u and v.
From Lemma 3.6 and Proposition 1.18:

$$\tilde{R}_{w,(1,n)}(t) = (1+t^2)\tilde{R}_{(n+1-b_1,n+1-b_2,\ldots,n+1-b_3,\ldots,n+1-b_{k-1},n+1-b_k),(1,n-1)}.$$

Let $u_1 = (n+1-b_1, n+1-b_2, \ldots, n+1-b_3, \ldots, n+1-b_{k-1}, n+1-b_k)$, since $d(u_1,(1,n-1)) = n-1$, by induction we have that $\tilde{R}_{u_1,(1,n-1)} = t^{k+2}(1+t^2)^{(inv((1,n-1)-inv(u_1)-(k+2))/2}$. It is easy to see that $inv(w) = inv(u_1) = 2(b_k - b_1) - (k-1)$ and since $inv((1,n)) = inv(1,n-1)+2$ the thesis follows.
Second proof
Since the pair $(w,(1,n))$ is under the conditions of Theorem 1.22, we have immediately that $\tilde{R}_{w,(1,n)}(t) = t^{k+2}(1+t^2)^{(inv(v)-inv(u)-(k+2))/2}$ and we have to apply 1.23 to compute a. We have to find a reduced expression for $w = (b_1, b_k, b_{k-1}, b_{k-2}, \ldots, b_2)$ which is a subexpression of $v = (1,n) = s_1 s_2 \ldots s_{n-2} s_{n-1} s_{n-2} \ldots s_2 s_1$. Recalling that a generic transposition (i,j) admits $s_i s_{i+1} \ldots s_{j-2} s_{j-1} s_{j-2} \ldots s_{i+1} s_i$ as reduced decomposition, we observe that being $w = (b_{k-1},b_k)(b_{k-2},b_{k-1})\ldots(b_3,b_4)(b_2,b_3)(b_1,b_2)$ then the juxtaposition of the reduced decomposition of these transposition gives a reduced decomposition for w, in fact:

$$w = s_{b_{k-1}} s_{b_{k-1}+1} \ldots s_{b_k-2} s_{b_k-1} s_{b_k-2} \ldots s_{b_{k-1}+1} s_{b_{k-1}} s_{b_{k-2}} s_{b_{k-2}+1} \ldots s_{b_{k-1}-2}$$
$$s_{b_{k-1}-1} s_{b_{k-1}-2} \ldots s_{b_{k-2}+1} s_{b_{k-2}} \ldots\ldots s_{b_2} s_{b_2+1} \ldots s_{b_3-2} s_{b_3-1} s_{b_3-2} \ldots s_{b_2+1} s_{b_2}$$
$$s_{b_1} s_{b_1+1} \ldots s_{b_2-2} s_{b_2-1} s_{b_2-2} \ldots s_{b_1+1} s_{b_1}$$

With the notation of Proposition 1.23 we have:

- $a_i = 0$, $\forall i \in [1, b_1-2] \cup [b_k, n-2]$, since $\tilde{w}(s_i) = 0$ while $\tilde{v}(s_i) = 2$, and s_i commutes with every s_j, $j > i$ such that $\tilde{w}(s_i) \neq 0$;
- $a_{b_1-1} = 2$, since $\tilde{w}(s_{b_1-1}) = 0$, $\tilde{v}(s_{b_1-1}) = 2$, but s_{b_1-1} does not commute with s_{b_1};
- $a_{b_i-1} = 1$, $\forall i \in [2,k]$ since $\tilde{v}(s_{b_i-1}) - \tilde{w}(s_{b_i-1}) = 2-1$;
- $a_{n-1} = 1$, since $\tilde{v}(s_{n-1}) = 1$.

Then $a = \sum a_i = k+2$, as we wanted to prove. □

Corollary 3.8. *Let $u \in S_n$ and $1 \leq i < a_1 < a_2 < a_3 < \cdots < a_k < j \leq n$, be such that:*

1. *u contains a pattern $k123\ldots k-1$ at the positions a_1, a_2, …, a_k, and $u < v = u(i,j)(a_1,a_2,a_3,\ldots,a_k)$*

2. *u contains a pattern $23\dots(k-1)k1$ at the positions $a_1, a_2, \dots, a_k$ and $u < v = u(i,j)(a_1, a_k, a_{k-1}, \dots, a_2)$*

Then

$$R_{u,v}(q) = (q-1)^{k+2}(q^2-q+1)^{(inv(v)-inv(u)-(k+2))/2}$$

Proof. It is an easy consequence of Corollary 3.7 and Theorem 1.16. □

In the following sections we apply these results to explicit some R-polynomials related to the partitions $(2,2,1,1,\dots)$, $(3,2,1,1,\dots)$ and $(4,2,1,1,\dots)$.

3.2 On the R-polynomials of $(2,2,1,1,\dots)$

As a consequence of Theorem 3.3 we have:

Corollary 3.9. Two nested transpositions *Let $u \in S_n$, $u(k) > u(l)$, $1 \le i < k < l < j \le n$ and suppose that $v = u(i,j)(k,l)$. Then*

$$\tilde{R}_{u,v}(t) = t^4(1+t^2)^{(inv(v)-inv(u)-4)/2}$$

Proof. This is the case of Theorem 3.3 for $k = 2$. □

Corollary 3.10. *Let $u \in S_n$, $u(k) > u(l)$, $1 \le i < k < l < j \le n$ and suppose that $v = u(i,j)(k,l)$. Then $R_{u,v}(q) = (q-1)^4(q^2-q+1)^{(inv(v)-inv(u)-4)/2}$*

The case of two disjoint transpositions has been proved in Corollary 2.13: $u, v \in S_n$ such that $v = u(i,j)(k,l)$ and $i < j < k < l$ then $\tilde{R}_{u,v}(t) = t^2(t^2+1)^{\frac{1}{2}(inv(v)-inv(u)-2)}$. We investigated also the remaining case: $v = u(i,j)(k,l)$ and $i < k < j < l$, i.e. the case in which the two transpositions are neither disjoint or nested. We made computations using Maple C.A.S. and so far we are not able to produce result on this case. It seems that does not exist a general product formula; we see the following example:

$$R_{(126543,436512)}(q) = (q^2-q+1)(q-1)^2$$
$$R_{(132456,562413)}(q) = (q^2-q+1)(q-1)^2(q^6-2q^5+2q^4-q^3+2q^2-2q+1)$$

In both cases we have applied to the smallest permutation the transposition $(1,5)(3,6)$.

3.3 On the R-polynomials of $(3, 2, 1, 1, \dots)$

We collect the result which follow from the previous section, on cycle type $(3, 2, 1, 1, \dots)$.

More precisely we consider permutations u which contain a pattern of type 231, 312 as applications of Theorem 3.3, and we prove that also the case of pattern 321 follows from these, by equivalence.
We also prove in 3.14 an explicit formulas for other irreducible classes depending on certain patterns of length 3 which derive from Corollary 3.9 and Theorem 3.12.
Before starting we recall that in Chapter 2, Corollary 2.15 we have already seen a class that belongs to $(3, 2, 1, 1, \dots)$, it was the case of pair (u, v) in which v is obtained by applying a 3-cycle and a disjoint transposition; now we examine cases of a 3-cycle nested to a transposition.

Lemma 3.11. *Let $u \in S_n$ and $1 \leq i < a < b < c < j \leq n$.*

1. *If $u(a) = max\{u(a), u(b), u(c)\}$ then*
$$u < v = u(1, n)(a, b, c) \Longleftrightarrow u(i) < min\{u(b), u(c)\} \quad \textit{and} \quad u(a) < u(j)$$
2. *If $u(c) = min\{u(a), u(b), u(c)\}$ then*
$$u < v = u(1, n)(a, c, b) \Longleftrightarrow u(i) < u(c) \quad \textit{and} \quad max\{u(a), u(b)\} < u(j)$$

Proof. The part 1) of the statement for the case that $u(a) > u(c) > u(b)$ has been proved in Lemma 3.2 with $k = 3$. We recall that

$$u = \dots u(i) \dots u(a) \dots u(b) \dots u(c) \dots u(j) \dots$$
$$v = \dots u(j) \dots u(b) \dots u(c) \dots u(a) \dots u(i) \dots$$

The case that $u(a) > u(b) > u(c)$, where u and v are as above, is identical but, in addition, we need to verify that $u(i) < u(c)$. Suppose that $u(c) < u(i)$ and consider the b-th rearrangement of u and v:

$$\{u(1), \dots, u(i), \dots u(a) \dots u(b)\} =$$
$$\{u^{b,1}, \dots, u^{b,s}, \mathbf{u(i)}, u^{b,s+2}, \dots, u^{b,r}, u(b), u^{b,r+2}, \dots, u^{b,t}, u(a), u^{b,t+2}, \dots\}_{\leq}$$
$$\{u(1), \dots, u(j), \dots u(b) \dots u(c)\} =$$
$$\{u^{b,1}, \dots, u(c), \dots, \mathbf{u^{b,s}}, u^{b,s+2}, \dots, u^{b,r}, u(b), u^{b,r+2}, \dots, u^{b,t} \dots, u(j), \dots\}_{\leq}$$

where $u^{b,s+1} = u(i) \not\leq v^{b,s+1} = u^{b,s}$. Hence we have proved that if $u < v$ then $u(i) < min\{u(b), u(c)\}$ and $u(a) < u(j)$ under the hypothesis that $u(a) = max\{u(a), u(b), u(c)\}$.

The part 2) follows as showed in Lemma 3.2 from the fact that the map $w \mapsto w^{-1}$ is an automorphism of Bruhat order.

□

Corollary 3.12. *Let* $u \in S_n$, $1 \leq i < a < b < c < j \leq n$ *and* $u(i) < u(s) < u(j), \forall s \in \{a, b, c\}$.

$$v = \begin{cases} u(i,j)(a,b,c), & \textit{if } u(a) = max\{u(a), u(b), u(c)\} \\ u(i,j)(a,c,b), & \textit{if } u(c) = min\{u(a), u(b), u(c)\} \end{cases}$$

Then

$$\tilde{R}_{u,v}(t) = t^5(1+t^2)^{(inv(v)-inv(u)-5)/2}$$

Proof. By Lemma 3.11 and Theorem 1.17 we can assume $i = 1 = u(1)$, and $j = n = u(n)$. The case that $u(b) < u(c) < u(a)$, i.e. u contains a pattern of type 312 at the positions a, b, c and $v = u(1,n)(a,b,c)$ is given by Theorem 3.3.
The case that $u(c) < u(a) < u(b)$, i.e. u contains a pattern of type 231 at the positions a, b, c and $v = u(1,n)(a,c,b)$ is given by Corollary 3.5.
Therefore we have to prove the cases in which $u(a) > u(b) > u(c)$, i.e u contains a pattern 321.
If $v = u(1,n)(a,c,b)$ then we have that (b,c) is a common inversion of u and v which behaves as a descent, by Lemma 2.10 In fact, we can suppose that the intervals $[2, a-1]$ and $[c+1, n-1]$ are fixed points and in $[a+1, b-1] \cup [c+1, n-1]$ there are not descents, moreover $u(b) = v(c)$: .

$$u = 12\ldots a-1u(a)u(a+1)\ldots u(b-1)u(b)u(b+1)\ldots u(c-1)u(c)c+1\ldots(n-1)n$$

$$v = n2\ldots a-1u(c)u(a+1)\ldots u(b-1)u(a)u(b+1)\ldots u(c-1)u(b)c+1\ldots(n-1)1$$

This implies that there exists $r \in [b, c-1]$ such that $(u,v) \sim (x,y)$, where $x = 12\ldots a-1u(a)u(a+1)\ldots u(b-1)u(b+1)\ldots u_r u(c)u(b)u_{r+1}c+1\ldots(n-1)n$ and $y = n2\ldots a-1u(c)u(a+1)\ldots u(b-1)u(b+1)\ldots u_r u(b)u(a)u_{r+1}\ldots c+1\ldots(n-1)1$; explicitly:

$$u = 12\ldots a{-}1u(a)u(a{+}1)\ldots u(b{-}1)\ldots u(r)u(c)u(b)u(r{+}1)\ldots u(c{-}1)c{+}1\ldots(n{-}1)n$$

$$v = n2\ldots a{-}1u(c)u(a{+}1)\ldots u(b{-}1)\ldots u(r)u(b)u(a)u(r{+}1)\ldots u(c{-}1)c{+}1\ldots(n{-}1)1$$

And x contains a pattern 312, and we are under the conditions of Theorem 3.3. If $v = u(1,n)(a,b,c)$ then (a,b) which is a common inversion of u and v behaves as a descent.

$$u = 12\ldots a-1u(a)u(a+1)\ldots u(b-1)u(b)u(b+1)\ldots u(c-1)u(c)c+1\ldots(n-1)n$$

$$v = n2\ldots a-1u(b)u(a+1)\ldots u(b-1)u(c)u(b+1)\ldots u(c-1)u(a)c+1\ldots(n-1)1$$

and since (a,b) is a common inversion of u and v behaves as a descent, by Lemma 2.10 then we reduce to the case of pattern 231 of Corollary 3.5. $\square$

Corollary 3.13. *Let $u \in S_n$, $1 \leq i < a < b < c < j \leq n$ and v under the conditions of Corollary 3.12. Then $R_{u,v}(q) = (q-1)^5(q^2-q+1)^{(inv(v)-inv(u)-5)/2}$.*

Proof. This follows from Theorem 1.16. $\square$

Remarks

The pairs of permutations (u, v) considered in Corollary 3.12 have all the property that $u^{-1}v = (1, n)x$ where x is a 3-cycle.

We observe that the hypotheses on $u(a), u(b), u(c)$ are the more general possible and that the fact that $u^{-1}v$ is in the conjugacy class of $(1, n)x$ is not sufficient to have the same R-polynomial:

$\tilde{R}_{12435,53241}(t) = t^5(2+t^2)$ and $12435 \circ 53241 = (1,5)(2,4,3)$.
Here it is $2 = u(a) \neq max\{2,3,4\}$ as in the next example
$\tilde{R}_{12345,54231}(t) = t^3(t^6+5t^4+6t^2+1)$ and $12345 \circ 54231 = (1,5)(2,4,3)$.

Therefore we can conclude that in the class of the cycle-type parametrized by the partition $(3, 2, 1, 1, \dots)$, is split by the R-polynomials in different equivalence classes of the relation $\sim$ defined in Chapter 2, as in the case of the partition $(2, 2, 1, 1, \dots)$.

Proposition 3.14. *Let $u, v \in S_n$ and $1 \leq i < a < b < c < j \leq n$ be such that one of the following condition is satisfied:*

i) u contains a pattern of type 132 at the positions a, b, c; $\forall s \in [a+1, b-1]$, $u(s) \notin [u(a), u(c)]$ and $v = u(i,j)(a,c,b)$

ii) u contains a pattern of type 312 at the positions a, b, c; $\forall s \in [b+1, c-1]$, $u(s) \notin [u(b), u(c)]$ and $v = u(i,j)(a,c,b)$

iii) u contains a pattern of type 213 at the positions a, b, c; $\forall s \in [b+1, c-1]$, $u(s) \notin [u(a), u(c)]$ and $v = u(i,j)(a,b,c)$

iv) u contains a pattern of type 231 at the positions a, b, c; $\forall s \in [a+1, b-1]$, $u(s) \notin [u(a), u(b)]$ and $v = u(i,j)(a,b,c)$

Then

$$\tilde{R}_{u,v}(t) = t^5(1+t^2)^{\frac{inv(v)-inv(u)-7}{2}}(2+t^2)$$

Proof. In order to lighten notation, we write all permutations, in their complete notation, only between the positions i, j.

$$\begin{array}{lccccccccccc} u = & \cdots & u(i) & \cdots & u(a) & \cdots & u(b) & \cdots & u(c) & \cdots & u(j) & \cdots \\ v = & \cdots & u(j) & \cdots & u(c) & \cdots & u(a) & \cdots & u(b) & \cdots & u(i) & \cdots . \end{array}$$

We prove part i).
From the hypotheses follows that $\forall s \in [a+1, b-1]$ either $u(s) < u(a)$ or $u(s) > u(c)$.

We assume that $u(a-1) < \cdots < u(b-1)$ and $u(b+1) < \cdots < u(c-1)$, by Theorem 1.17. Define $t \stackrel{\text{def}}{=} max\{m \in [a+1, b-1] : u(m) < u(a)\}$, it follows that $u(a+1) < \cdots < u(t) < u(a) < u(c) < u(t+1)$.

Hence $(u, v) \sim (u_1, v_1)$ where $u_1 = us_a s_{a+1} \ldots s_t$, $u_1 = us_a s_{a+1} \ldots s_t$. More explicitly

$$\begin{array}{ll} u_1 & = u(i) \\ v_1 & = u(j) \end{array} \cdots \begin{array}{l} u(a-1)u(a+1) \\ u(a-1)u(a+1) \end{array} \cdots \begin{array}{l} u(t)u(a)u(t+1) \\ u(t)u(c)u(t+1) \end{array} \cdots \begin{array}{l} u(b-1)u(b)u(b+1) \\ u(b-1)u(a)u(b+1) \end{array} \cdots$$

$$\begin{array}{l} u(c-1)u(c)u(c+1) \\ u(c-1)u(b)u(c+1) \end{array} \cdots \begin{array}{l} u(j) \\ u(i) \end{array}$$

We reverse these permutations:

$$\begin{array}{ll} v_1 w_0 & = u(i) \\ u_1 w_0 & = u(j) \end{array} \cdots \begin{array}{l} u(c+1)u(b)u(c-1) \\ u(c+1)u(c)u(c-1) \end{array} \cdots \begin{array}{l} u(b+1)u(a)u(b-1) \\ u(b+1)u(b)u(b-1) \end{array} \cdots \begin{array}{l} u(t+1)\, u(c)\, u(t) \\ u(t+1)\, u(a)\, u(t) \end{array}$$

$$\cdots \begin{array}{l} u(a+1)u(a-1) \\ u(a+1)u(a-1) \end{array} \cdots \begin{array}{l} u(j) \\ u(i) \end{array}$$

By definition of t we have that $(v_1 w_0, u_1 w_0)$ is equivalent to the next pair:

$$\begin{array}{ll} u_2 & = u(i) \\ v_2 & = u(j) \end{array} \cdots \begin{array}{l} u(c+1)u(b)u(c-1) \\ u(c+1)u(c)u(c-1) \end{array} \cdots \begin{array}{l} u(b+1)u(a)u(c)u(b-1) \\ u(b+1)u(b)u(a)u(b-1) \end{array} \cdots \begin{array}{l} u(a+1)u(a-1) \\ u(a+1)u(a-1) \end{array} \cdots \begin{array}{l} u(j) \\ u(i) \end{array}$$

We reverse again to obtain:

$$\begin{array}{ll} u_3 & = u(i) \\ v_3 & = u(j) \end{array} \cdots \begin{array}{l} u(a-1)u(a+1)\, u(b-1)u(a)u(b)u(b+1) \\ u(a-1)u(a+1)\, u(b-1)u(c)u(a)u(b+1) \end{array} \cdots \begin{array}{l} u(c-1)u(c)u(c+1) \\ u(c-1)u(b)u(c+1) \end{array} \cdots \begin{array}{l} u(j) \\ u(i) \end{array}$$

Here we have defined $u_3 = v_2 w_0$ and $v_3 = u_2 w_0$.

Now we compute $\tilde{R}_{u_3,v_3}(t)$ by applying Theorem 1.17 to the descent $s_{b-1} \in D(v_3)$; since $s_{b-1} \notin D(u_3)$ we have: $\tilde{R}_{u_3,v_3}(t) = \tilde{R}_{u_3 s_{b-1}, v_3 s_{b-1}}(t) + t\tilde{R}_{u_3, v_3 s_{b-1}}(t)$ The pair $(u_3 s_{b-1}, v_3 s_{b-1})$ satisfies the hypotheses of Corollary 3.12, in fact $u_3 s_{b-1}$ contains a pattern 312 at the positions $b-1, b, c$ and $v_3 s_{b-1} = u_3 s_{b-1}(i,j)(b-1, b, c)$; whereas the pair $(u_3, v_3 s_{b-1})$ is under the conditions of Corollary 3.9, since $u_3(b-1) > u_3(b)$ and $v_3 s_{b-1} = u_3(i,j)(b-1, b)$.
Therefore

$$\tilde{R}_{u_3,v_3}(t) = t^5(1+t^2)^{\frac{inv(v_3 s_{b-1})-inv(u_3 s_{b-1})-5}{2}} + t(t^4(1+t^2)^{\frac{inv(v_3 s_{b-1})-inv(u_3)-4}{2}}) =$$

$$t^5(1+t^2)^{\frac{inv(v_3)-inv(u_3)-7}{2}}(2+t^2)$$

The last equality follows from $inv(u_3) = inv(u_3 s_{b-1}) - 1$ and $inv(v_3) = inv(u_3 s_{b-1}) + 1$, since s_{b-1} is a descent of v_3, but not of u_3. The final result follows from the fact that $(u,v) \sim (u_3,v_3)$ implies that $inv(v) - inv(u) = inv(v_3) - inv(u_3)$.

Now we consider to be under the conditions ii).
We assume that the values between the positions $b+1$ and $c-1$ are in increasing order. Since $\forall s \in [b+1, c-1]$, $u(s) \notin [u(b), u(c)]$ we have that either $u(s) < u(b)$ or $u(s) > u(c)$. Let $t \stackrel{\text{def}}{=} max\{m \in [b+1, c-1] : u(m) < u(b)\}$, it follows that $u(b+1) < \cdots < u(t) < u(b) < u(c) < u(t+1)$. We have that $(u,v) \sim (u_1, v_1)$ where $u_1 = us_{c-1} \ldots s_{t+1}$, $u_1 = us_{c-1} \ldots s_{t+1}$.

$$\begin{matrix} u_1 = u(i) & \cdots & u(a) & \cdots & u(b)u(b+1) & \cdots & u(t)u(c)u(t+1) & \cdots & u(j) \\ v_1 = u(j) & & u(c) & & u(a)u(b+1) & & u(t)u(b)u(t+1) & & u(i) \end{matrix}$$

This pair is equivalent to the next one:

$$\begin{matrix} v_1 w_0 = u(i) & \cdots & u(t+1)u(b)u(t) & \cdots & u(b+1)u(a) & \cdots & u(c) & \cdots & u(j) \\ u_1 w_0 = u(j) & & u(t+1)u(c)u(t) & & u(b+1)u(b) & & u(a) & & u(i) \end{matrix}$$

And again by definition of t we can move the column $\begin{matrix} u(b) \\ u(c) \end{matrix}$ to the right to obtain:

$$\begin{matrix} u_2 = u(i) & \cdots & u(t+1)u(t) & \cdots & u(b+1)u(b)u(a) & \cdots & u(c) & \cdots & u(j) \\ v_2 = u(j) & & u(t+1)u(t) & & u(b+1)u(c)u(b) & & u(a) & & u(i) \end{matrix}$$

We reverse:

$$\begin{matrix} u_3 = u(i) & \cdots & u(a) & \cdots & u(b)u(c)u(b+1) & \cdots & u(j-1)u(j) \\ v_3 = u(j) & & u(c) & & u(a)u(b)u(b+1) & & u(j-1)u(i) \end{matrix}$$

At this point if we apply Theorem 1.17 to the descent $(b-1, b) \in D(v_3)$ we obtain two summands which are under the conditions of Theorem 3.9 and 3.12 as in i).

We prove case iii).
As before we let $t \stackrel{\text{def}}{=} max\{m \in [b+1, c-1] : u(m) < u(a)\}$, it follows that $u(b+1) < \cdots < u(t) < u(a) < u(c) < u(t+1)$.

$$\begin{matrix} u = u(i)u(i+1) & \cdots & u(a) & \cdots & u(b) & \cdots & u(t)\,u(t+1) & \cdots & u(c) & \cdots & u(j-1)u(j) \\ v = u(j)u(i+1) & & u(b) & & u(c) & & u(t)\,u(t+1) & & u(a) & & u(j-1)u(i) \end{matrix}$$

With the same computation of the previous cases, we obtain the pair (u_1, v_1), which is explicitly:

$$\begin{array}{lcccccccc} u_1 = u(i)u(i+1) & \multirow{2}{*}{$\cdots$} & u(a) & \multirow{2}{*}{$\cdots$} & u(b)\,u(b+1) & \multirow{2}{*}{$\cdots$} & u(t)\,u(c)\,u(t+1) & \multirow{2}{*}{$\cdots$} & u(j-1)u(j) \\ v_1 = u(j)u(i+1) & & u(b) & & u(c)\,u(b+1) & & u(t)\,u(a)\,u(t+1) & & u(j-1)u(i) \end{array}$$

Then we multiply on the right by w_0:

$$\begin{array}{lcccccccc} v_1 w_0 = u(i)u(j-1) & \cdots & u(t+1)\,u(a)\,u(t) & \cdots & u(b+1)\,u(c) & \cdots & u(b) & \cdots & u(i+1)u(j) \\ u_1 w_0 = u(j)u(j-1) & \cdots & u(t+1)\,u(c)\,u(t) & \cdots & u(b+1)\,u(b) & \cdots & u(a) & \cdots & u(i+1)u(i) \end{array}$$

By definition of t, we can move the column $\begin{array}{c} u(a) \\ u(c) \end{array}$ and it results:

$$\begin{array}{lcccccccc} u_2 = u(i)u(j-1) & \cdots & u(t+1)\,u(t) & \cdots & u(b+1)\,u(a)\,u(c) & \cdots & u(b) & \cdots & u(i+1)u(j) \\ v_2 = u(j)u(j-1) & \cdots & u(t+1)\,u(t) & \cdots & u(b+1)\,u(c)\,u(b) & \cdots & u(a) & \cdots & u(i+1)u(i) \end{array}$$

Now define $u_3 = v_2 w_0$ and $v_3 = u_2 w_0$. To end the computation we apply again Theorem 1.17 to the descent s_{b-1} of v_3; then we split the $\tilde{R}_{u_3,v_3}(t)$ in two summands which are $\tilde{R}_{u_3,v_3 s}$ that satisfies the hypotheses of Theorem 3.9, since $u_3(b-1) > u_3(b)$ and $v_3 = u_3(i,j)(b-1,b)$ and $\tilde{R}_{u_3 s, v_3 s}$ that is under the conditions of Theorem 3.12, being $u_3 s_{b-1}$ contains a pattern 231 at the positions $a, b-1, b$ and $v_3 s_{b-1} = u_3 s_{b-1}(i,j)(a, b-1, b)$.

Finally, we treat case iv).
To prove this case it is enough to observe that if (u, v) satisfies iv) then (vw_0, uw_0) satisfies the conditions of iii). In fact, if we define $u_1 = vw_0$, $v_1 = uw_0$, $\bar{a} = u_1^{-1}(u(a))$, $\bar{b} = u_1^{-1}(u(c))$ and $\bar{c} = u_1^{-1}(u(b))$, then $\forall s \in [\bar{b}+1, \bar{c}-1]$, $u_1(s) \notin [u_1(\bar{a}), u_1(\bar{c}]$, since $u_1(\bar{a}) = u(a)$ and $u_1(\bar{c}) = u(b)$. For an easier visualization we write explicitly u_1, v_1:

$$\begin{array}{lcccccccc} u_1 = u(i)u(j-1) & \cdots & u(a) & \cdots & u(c) & \cdots & u(c) & \cdots & u(i+1)u(j) \\ v_1 = u(j)u(j-1) & \cdots & u(c) & \cdots & u(b) & \cdots & u(a) & \cdots & u(i+1)u(i) \end{array}$$

and it is clear that u_1 contains a pattern 213 at the positions $\bar{a}, \bar{b}, \bar{c}$.
This concludes the proof $\square$

Corollary 3.15. *Let $u, v \in S_n$ and $1 \leq i < a < b < c < j \leq n$ be such that one of the following condition is satisfied:*

i) u contains a pattern of type 132 at the positions a, b, c; $\forall s \in [a+1, b-1]$, $u(s) \notin [u(a), u(c)]$ and $v = u(i,j)(a,c,b)$

ii) *u contains a pattern of type* 312 *at the positions* a, b, c; $\forall s \in [b+1, c-1]$, $u(s) \notin [u(b), u(c)]$ *and* $v = u(i,j)(a,c,b)$

iii) *u contains a pattern of type* 213 *at the positions* a, b, c; $\forall s \in [b+1, c-1]$, $u(s) \notin [u(a), u(c)]$ *and* $v = u(i,j)(a,b,c)$

iv) *u contains a pattern of type* 231 *at the positions* a, b, c; $\forall s \in [a+1, b-1]$, $u(s) \notin [u(a), u(b)]$ *and* $v = u(i,j)(a,b,c)$

Then, if $u < v$

$$R_{u,v}(q) = (q-1)^5 (q^2 - q + 1)^{\frac{inv(v) - inv(u) - 7}{2}} (q^2 + 1)$$

3.4 On the R-polynomials of (4,2,1,1,...)

We consider irreducible pairs (u, v) in which u contains a pattern of length 4 between two fixed positions i, j, and such that $u^{-1}v$ is of cycle type $(4, 2, 1, 1, \dots)$. The classes given in Theorem 3.17 satisfy the rule of natural reordering of a pattern, i.e. we have a pattern of length 4 in u, and we apply to u the transposition (i, j) and a 4-cycle, nested to $(, j)$, which put these elements in their natural ordering. If the pattern is 4123 or 2341, the closed product formula for $R_{u,v}(t)$ follows from Theorem 3.3 for $k = 4$, while the others follow by equivalence with these cases. Before proving the theorem we give a characterization of the Bruhat order, for these pairs.

Lemma 3.16. *Let* (u, v), $u, v \in S_n$, $1 \le i < a_1 < a_2 < a_3 < a_4 < j \le n$, *be one of the following pairs of permutations:*
At the positions a_1, a_2, a_3, a_4,

1. *u contains a pattern of type* 4123 *and* $v = u(i,j)(a_1, a_2, a_3, a_4)$;
2. *u contains a pattern of type* 2341 *and* $v = u(i,j)(a_1, a_4, a_3, a_2)$;
3. *u contains a pattern of type* 2413 *and* $v = u(i,j)(a_1, a_3, a_4, a_2)$;
4. *u contains a pattern of type* 3142 *and* $v = u(i,j)(a_1, a_2, a_4, a_3)$.

Then $u < v \iff u(i) < min\{u(a_s) : s \in [4]\}$ *and* $u(j) > max\{u(a_s) : s \in [4]\}$

Proof. The statement for the first two classes is a particular case of Lemma 3.2.
We consider u that contains the pattern 2413 and $v = u(i,j)(a_1, a_3, a_4, a_2)$, explicitly:

$$u = 1 \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(a_4) \dots 1$$
$$v = n \dots u(a_3) \dots u(a_1) \dots u(a_4) \dots u(a_2) \dots n$$

We apply Theorem 1.9. As in 3.2 we have immediately that $u(i) < u(j)$, we have to prove that $u(i) < u(a_3)$ and $u(a_2) < u(j)$.
At first we prove that $u(a_1) > u(i)$, in fact if $u(a_1) < u(i)$ then we consider the a_1-rearrangement for u and v:

$$\{u(1), \ldots, u(i), \ldots, u(a_1)\} = \{u^{a_1,1}, \ldots, u^{a_1,t}, \mathbf{u(a_1)}, u^{a_1,t+2}, \ldots, u(i), \ldots\}_{\leq}$$

$$\{u(1), \ldots, u(j), \ldots u(a_3)\} = \{u^{a_1,1}, \ldots, u(a_3), \ldots, \mathbf{u^{a_1,t}}, u^{a_1,t+2}, \ldots, u(j), \ldots\}_{\leq},$$

where $u^{a_1,t+1}$ is the position occupied by $u(a_1)$ in the a_1-th rearrangement. It follows that $u \not\leq v$, since $u(a_1) = u^{a_1,t+1} \not\leq v^{a_1,t+1} = u^{a_1,t}$.
Suppose that $u(i) > u(a_3)$ then the a_1-th rearrangement is:

$$\{u(1), \ldots, u(i), \ldots, u(a_1)\} = \{u^{a_1,1}, \ldots, u^{a_1,s}, \mathbf{u(i)}, u^{a_1,s+2}, \ldots, u(a_1), \ldots\}_{\leq}$$

$$\{u(1), \ldots, u(j), \ldots, u(a_3)\} = \{u^{a_1,1}, \ldots, u(a_3), \ldots, \mathbf{u^{a_1,s}}, u^{a_1,s+2}, \ldots, u(j), \ldots\}_{\leq}$$

Note that $u(i) = u^{a_1,s+1} \not\leq v^{a_1,s+1} = u^{a_1,s}$, so that $u \not\leq v$.
So far we have proved that $u(i) < u(a_3) < u(a_1) < u(j)$, it remains to show that $u(a_2) < u(j)$. If it is not, i.e. $u(a_2) > u(j)$ we consider the a_2-rearrangement

$$\{u(1), \ldots, u(i), \ldots, u(a_1), \ldots, u(a_2)\} =$$
$$\{u^{a_2,1}, \ldots, u(i), \ldots, u(a_1), \ldots, u^{a_2,t}, \mathbf{u(a_2)}, u^{a_2,t+2}, \ldots\}_{\leq}$$
$$\{u(1), \ldots, u(j), \ldots, u(a_3), \ldots, u(a_1)\} =$$
$$\{u^{a_2,1}, \ldots, u(a_3), \ldots, u(a_1), \ldots, u(j), \ldots, \mathbf{u^{a_2,t}}, u^{a_2,t+2}, \ldots\}_{\leq}$$

then $u(a_2, t+1) = u(a_2) \not\leq v(a_2, t+1) = u(a_2, t)$, and this concludes the proof of 3.

Let u be under the conditions of 4., i.e. contains a pattern 3142 and $v = u(i,j)(a_1, a_2, a_4, a_3)$, we show that (u^{-1}, v^{-1}) is of type 3. and then since the Bruhat order is preserved by the inversion we will have the thesis.

$$u = u(i) \ldots u(a_1) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots u(j)$$
$$v = u(j) \ldots u(a_2) \ldots u(a_4) \ldots u(a_1) \ldots u(a_3) \ldots u(i)$$

And then since in (u^{-1}, v^{-1}) the central pattern corresponds to the one of the pair of type 3:

$$u^{-1} = \ldots a_2 \ldots a_4 \ldots a_1 \ldots a_3 \ldots$$
$$v^{-1} = \ldots a_1 \ldots a_2 \ldots a_3 \ldots a_4 \ldots$$

the conditions previously determined imply that the positions of the columns $\begin{smallmatrix} i \\ \cdot \\ j \end{smallmatrix}$ and $\begin{smallmatrix} j \\ \cdot \\ i \end{smallmatrix}$ have to be as follows:

$$u^{-1} = i \ldots a_2 \ldots a_4 \ldots a_1 \ldots a_3 \ldots j$$
$$v^{-1} = j \ldots a_1 \ldots a_2 \ldots a_3 \ldots a_4 \ldots i$$

$\square$

Theorem 3.17. *Let (u,v), $u,v \in S_n$, $1 \leq i < a_1 < a_2 < a_3 < a_4 < j \leq n$, be one of the following pairs of permutations:*
At the positions a_1, a_2, a_3, a_4,

1. *u contains a pattern of type 4123 and $v = u(i,j)(a_1,a_2,a_3,a_4)$;*
2. *u contains a pattern of type 2341 and $v = u(i,j)(a_1,a_4,a_3,a_2)$;*
3. *u contains a pattern of type 2413 and $v = u(i,j)(a_1,a_3,a_4,a_2)$;*
4. *u contains a pattern of type 3142 and $v = u(i,j)(a_1,a_2,a_4,a_3)$.*

If $u < v$, then

$$\tilde{R}_{u,v}(t) = t^6(1+t^2)^{(inv(v)-inv(u)-6)/2}$$

Proof. The cases 1) and 2) are Theorem 3.3 and Corollary 3.5 for $k = 4$.

We prove 3), 4) by showing that these two classes of permutations belong to the equivalence class described in 2). We can assume $i = 1$ and $j = n$ and $u(1) = 1$, $u(n) = n$, by Lemma 3.16 and Theorem 1.17; then we write u, v only in the positions in which they differ.
Let u be under the conditions 3), i.e. u contains a pattern 2413 at the positions a_1, a_2, a_3, a_4, with $1 < a_1 < a_2 < a_3 < a_4 < n$.

$$\begin{aligned} u &= 1 \ldots u(a_1) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots 1 \\ v &= n \ldots u(a_3) \ldots u(a_1) \ldots u(a_4) \ldots u(a_2) \ldots n \end{aligned}$$

Note that by hypothesis $u(a_3) < u(a_1) < u(a_4) < u(a_2)$ and $v = u(1,n)(a_1,a_2,a_4,a_3)$. We reverse these permutations and obtain:

$$\begin{aligned} vw_0 &= 1 \ldots u(a_2) \ldots u(a_4) \ldots \mathbf{u(a_1)} \ldots \mathbf{u(a_3)} \ldots 1 \\ uw_0 &= n \ldots u(a_4) \ldots u(a_3) \ldots \mathbf{u(a_2)} \ldots \mathbf{u(a_1)} \ldots n \end{aligned}$$

We can consider that the elements of the common blocks are in increasing order, and we apply Lemma 2.10 to the common inversion (k_1, k_2), with $k_1 = (vw_0)^{-1}(u(a_1)) = (uw_0)^{-1}(u(a_2))$ and $k_2 = (vw_0)^{-1}(u(a_3)) = (uw_0)^{-1}(u(a_1))$. Since (k_1, k_2) behaves as a descent then we have that (u,v) is equivalent to the following pair:

$$\begin{aligned} u_1 &= 1 \ldots u(a_2) \ldots u(a_4) \ldots \mathbf{u(a_3)}\, \mathbf{u(a_1)} \ldots 1 \\ v_1 &= n \ldots u(a_4) \ldots u(a_3) \ldots \mathbf{u(a_1)}\, \mathbf{u(a_2)} \ldots n \end{aligned}$$

For simplicity we let $b_1, b_2, b_3 \in [2, n-2]$ be such that $u_1(b_1) = u(a_2) = v_1(b_4)$, $u_1(b_2) = u(a_4) = v_1(b_1)$, $u_1(b_3) = u(a_3) = v_1(b_2)$ and $u_1(b_3+1) = u(a_1) = v_1(b_3)$, then $u_1(b_3) < u_1(b_3+1) < u_1(b_2) < u_1(b_1)$ and we rewrite (u_1, v_1):

$$\begin{array}{l} u_1 = 1 \dots u_1(b_1) \dots u_1(b_2) \dots \quad u_1(b_3) \quad u_1(b_3+1) \dots 1 \\ v_1 = n \dots u_1(b_2) \dots u_1(b_3) \dots u_1(b_3+1) \quad u_1(b_1) \quad \dots n \end{array}$$

$$\begin{array}{l} u_1^{-1} = 1 \dots b_3 \dots \boxed{b_3+1} \dots \boxed{b_2} \dots b_1 \dots 1 \\ v_1^{-1} = n \dots b_2 \dots \boxed{b_3} \dots \boxed{b_1} \dots b_4 \dots n \end{array}$$

Then we apply Lemma 2.10 to the inversion between the boxed columns and we have that $(u_1, v_1) \sim (u_2, v_2)$, where

$$\begin{array}{l} u_2 = 1 \dots \boxed{b_3} \dots \boxed{b_2}\, b_3+1 \dots \quad b_1 \quad \dots 1 \\ v_2 = n \dots \boxed{b_2} \dots \boxed{b_1} \quad b_3 \quad \dots b_3+1 \dots n \end{array}$$

And then again by Lemma 2.10 applied to the inversion between the boxed columns of (u_2, v_2) we have the next pair:

$$\begin{array}{l} u_3 = 1 \dots b_2\, b_3 \dots b_3+1 \dots \quad b_1 \quad \dots 1 \\ v_3 = n \dots b_1\, b_2 \dots \quad b_3 \quad \dots b_3+1 \dots n \end{array}$$

We have proved that $(u, v) \sim (u_3, v_3)$ where (u_3, v_3) is a pair of type 2).

To prove 4) it is enough to observe that a pair under these conditions is equivalent to a pair of type 3), by inversion. In fact

$$\begin{array}{l} u = 1 \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(a_4) \dots 1 \\ v = n \dots u(a_2) \dots u(a_4) \dots u(a_1) \dots u(a_3) \dots n \end{array}$$

Then

$$\begin{array}{l} u^{-1} = 1 \dots a_3 \dots a_1 \dots a_4 \dots a_2 \dots 1 \\ v^{-1} = n \dots a_1 \dots a_2 \dots a_3 \dots a_4 \dots n \end{array}$$

is of type 4) and this concludes the proof.

□

We give the correspondent result on the R-polynomials:

Corollary 3.18. *Let (u, v), $u, v \in S_n$, $1 \le i < a_1 < a_2 < a_3 < a_4 < j \le n$, be one of the following pairs of permutations:*
at the positions a_1, a_2, a_3, a_4,

1. *u contains a pattern of type 4123 and $v = u(i, j)(a_1, a_2, a_3, a_4)$;*
2. *u contains a pattern of type 2341 and $v = u(i, j)(a_1, a_4, a_3, a_2)$;*
3. *u contains a pattern of type 2413 and $v = u(i, j)(a_1, a_2, a_4, a_3)$;*

4. u contains a pattern of type 3142 *and* $v = u(i,j)(a_1,a_2,a_4,a_3)$.

If $u < v$, *then*

$$R_{u,v}(q) = (q-1)^6(q^2-q+1)^{(inv(v)-inv(u)-6)/2}$$

We consider now permutations u which contain a decreasing subsequence of length 4, i.e. a pattern of type 4321 at the positions a_1, a_2, a_3, a_4, where $1 \leq i < a_1 < a_2 < a_3 < a_4 < j \leq n$; we prove that if $u < v = u(i,j)x$, where x is a 4-cycle on a_1, a_2, a_3, a_4, then it is an easy consequence of the above proposition that $R_{u,v}(q) = (q-1)^6(q^2-q+1)^{(inv(v)-inv(u)-6)/2}$:

Corollary 3.19. *Let* (u,v), $u, v \in S_n$, $1 \leq i < a_1 < a_2 < a_3 < a_4 < j <\leq n$, *be such that* u *contains a pattern* 4321 *at the positions* a_1, a_2, a_3, a_4, *and* $v = u(i,j)x$, *where* x *is a 4 cycle which moves* a_1, a_2, a_3, a_4. *Then*

$$\begin{cases} (u,v) \sim (\alpha,\beta), & \textit{if} \quad x \in \{(a_1,a_2,a_3,a_4),(a_1,a_4,a_3,a_2),(a_1,a_3,a_4,a_2),(a_1,a_2,a_4,a_3)\} \\ R_{u,v}(q) = 0, & \textit{if} \quad x \in \{(a_1,a_3,a_2,a_4),(a_1,a_4,a_2,a_3)\} \end{cases}$$

Proof. The equivalence comes from the multiplication to the right by w_0, in each case.
We will see that the conditions on the bruhat ordering forces that
$x \notin \{(a_1,a_3,a_2,a_4),(a_1,a_4,a_2,a_3)\}$ to be $u < v$.
Consider u such that $u(a_1) > u(a_2) > u(a_3) > u(a_4)$ and $v = u(i,j)x$:

$$\begin{aligned} u &= \ldots u(i) \ldots u(a_1) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots u(j) \ldots \\ v &= \ldots u(j) \ldots v(a_1) \ldots v(a_2) \ldots v(a_3) \ldots v(a_4) \ldots u(i) \ldots \end{aligned}$$

Since we apply a 4-cycle, x which moves the elements of the pattern, we have that $v(a_1) \in \{u(a_2), u(a_3), u(a_4)\}$. Therefore $v(a_1) < u(a_1)$. By Theorem 1.9 it is $u(i) < u(j)$, as in Lemma 3.16 it can be proved that $u(i) < v(a_1) < u(a_1) < u(j)$

- $v(a_1) = u(a_2)$ this implies that $v(a_2) \in \{u(a_3), u(a_4)\}$ (if $v(a_2) = u(a_1)$ then x is not a 4-cycles).
  We consider $v(a_2) = u(a_3)$ this implies that $v(a_3) = u(a_4)$ and $v(a_4) = u(a_1)$,

  $$\begin{aligned} u &= \ldots u(i) \ldots u(a_1) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots u(j) \ldots \\ v &= \ldots u(j) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots u(a_1) \ldots u(i) \ldots \end{aligned}$$

  and by reversing we have:

  $$\begin{aligned} vw_0 &= \ldots u(i) \ldots u(a_1) \ldots u(a_4) \ldots u(a_3) \ldots u(a_2) \ldots u(j) \ldots \\ uw_0 &= \ldots u(j) \ldots u(a_4) \ldots u(a_3) \ldots u(a_2) \ldots u(a_1) \ldots u(i) \ldots \end{aligned}$$

which is a pair of type 1. of Theorem 3.17.
If $v(a_2) = u(a_4)$ then

$$\begin{aligned} u &= \ldots u(i) \ldots u(a_1) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots u(j) \ldots \\ v &= \ldots u(j) \ldots u(a_2) \ldots u(a_4) \ldots u(a_1) \ldots u(a_3) \ldots u(i) \ldots \end{aligned}$$

and by reversing we have:

$$\begin{aligned} vw_0 &= \ldots u(i) \ldots u(a_3) \ldots u(a_1) \ldots u(a_4) \ldots u(a_2) \ldots u(j) \ldots \\ uw_0 &= \ldots u(j) \ldots u(a_4) \ldots u(a_3) \ldots u(a_2) \ldots u(a_1) \ldots u(i) \ldots \end{aligned}$$

which is a pair of type 3. of Theorem 3.17.

- $v(a_1) = u(a_3)$ this implies that $v(a_2) \in \{u(a_1), u(a_4)\}$ If $v(a_2) = u(a_1)$ then:

$$\begin{aligned} u &= \ldots u(i) \ldots u(a_1) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots u(j) \ldots \\ v &= \ldots u(j) \ldots u(a_3) \ldots u(a_1) \ldots u(a_4) \ldots u(a_2) \ldots u(i) \ldots \end{aligned}$$

and by reversing we have:

$$\begin{aligned} vw_0 &= \ldots u(i) \ldots u(a_2) \ldots u(a_4) \ldots u(a_1) \ldots u(a_3) \ldots u(j) \ldots \\ uw_0 &= \ldots u(j) \ldots u(a_4) \ldots u(a_3) \ldots u(a_2) \ldots u(a_1) \ldots u(i) \ldots \end{aligned}$$

which is a pair of type 4. of Theorem 3.17.

If $v(a_2) = u(a_4)$ then:

$$\begin{aligned} u &= \ldots u(i) \ldots u(a_1) \ldots u(a_2) \ldots u(a_3) \ldots u(a_4) \ldots u(j) \ldots \\ v &= \ldots u(j) \ldots u(a_3) \ldots u(a_4) \ldots u(a_2) \ldots u(a_1) \ldots u(i) \ldots \end{aligned}$$

In this case we have $u \not\leq v$ in fact by Theorem 1.9, if we consider the a_2 rearrangement, we have:

$$\{u(1), \ldots, u(i), \ldots, u(a_1), \ldots, u(a_2)\} =$$

$$\{u^{a_2,1}, \ldots, u(i), \ldots, u^{a_2,t}, \mathbf{u(a_2)}, u^{a_2,t+2}, \ldots, , u^{a_2,s+t}, u(a_1), , u^{a_2,s+t+2}, \}_{\leq}$$

$$\{u(1), \ldots, u(j), \ldots, u(a_3), \ldots, u(a_4)\} =$$

$$\{u^{a_2,1}, \ldots, u(4), \ldots, u(a_3), \ldots, \mathbf{u^{a_2,t}}, u^{a_2,t+2}, \ldots\}_{\leq}$$

then $u(a_2, t+1) = u(a_2) \not\leq v(a_2, t+1) = u(a_2, t)$.

- $v(a_1) = u(a_4)$ this implies that $v(a_2) \in \{u(a_1), u(a_3)\}$.
  If $v(a_2) = u(a_1)$ then $v(a_3) = u(a_4)$ and $v(a_4) = u(a_3)$:

$$u = \dots u(i) \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(a_4) \dots u(j) \dots$$
$$v = \dots u(j) \dots u(a_4) \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(i) \dots$$

  and this is equivalent to a pair of type 2. of Theorem 3.17, in fact:

$$vw_0 = \dots u(i) \dots u(a_3) \dots u(a_2) \dots u(a_2) \dots u(a_4) \dots u(j) \dots$$
$$uw_0 = \dots u(j) \dots u(a_4) \dots u(a_3) \dots u(a_1) \dots u(a_1) \dots u(i) \dots$$

  If $v(a_2) = u(a_3)$ then

$$u = \dots u(i) \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(a_4) \dots u(j) \dots$$
$$v = \dots u(j) \dots u(a_4) \dots u(a_3) \dots u(a_1) \dots u(a_2) \dots u(i) \dots$$

  and by Theorem 1.9 $u \not\leq v$.

Summarizing we have proved that if $v = u(i,j)(a_1, a_3, a_2, a_4)$ or $v = u(i,j)(a_1, a_4, a_2, a_3)$, then $u \not\leq v$ and then $R_{u,v}(q) = 0$, while in the other cases we obtain a pair equivalent to one of Theorem 3.17.

□

Here we have other patterns which are related to the ones above:

Proposition 3.20. *Let (u,v), $u, v \in S_n$, $1 \leq i < a_1 < a_2 < a_3 < a_4 < j \leq n$, be one of the following pairs of permutations:*
At the positions a_1, a_2, a_3, a_4,

1. *u contains a pattern of type 4312 and $v = u(i,j)(a_1, a_2, a_3, a_4)$;*

2. *u contains a pattern of type 3421 and $v = u(i,j)(a_1, a_4, a_3, a_2)$.*

Then (u,v) is equivalent to the pair of Theorem 3.17

Proof. If u contains a pattern of type 4312 between the positions i,j this means, by definition, that $u(i) < u(a_3) < u(a_4) < u(a_2) < u(a_1) < u(j)$ and $v = u(i,j)(a_1, a_2, a_3, a_4)$:

$$u = \dots u(i) \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(a_4) \dots u(j) \dots$$
$$v = \dots u(j) \dots u(a_2) \dots u(a_3) \dots u(a_4) \dots u(a_1) \dots u(i) \dots$$

We have that $(a_1, a_2) \in Inv(u) \cap Inv(v)$ and we apply Lemma 2.10: there exists $r \in [a+1, b-1]$ such that $(u,v) \sim (\sigma, \tau)$, where σ and τ are the following permutations:

$$\sigma = u(i) \dots u(a_1+1) \dots u(r)\, u(a_2)\, u(a_1)\, u(r+1) \dots u(a_3) \dots u(a_4) \dots u(j)$$
$$\tau = v(j) \dots v(a_1+1) \dots u(r)\, u(a_3)\, u(a_2)\, u(r+1) \dots u(a_4) \dots u(a_1) \dots u(i)$$

Now $(r+2,c) \in Inv(\sigma) \cap Inv(\tau)$ and again Lemma 2.10 can be applied. This implies that there exists $s \in [r+3, c-1]$ such that $(\sigma, \tau) \sim (\alpha, \beta)$, where

$$\alpha = u(i) \dots u(a_1+1) \dots u(r)\, u(a_2)\, u(r+1) \dots u(s)\, u(a_3)\, u(a_1)\, u(s+1) \dots u(a_4) \dots u(j)$$
$$\beta = u(j) \dots u(a_1+1) \dots u(r)\, u(a_3)\, u(r+1) \dots u(s)\, u(a_4)\, u(a_2)\, u(s+1) \dots u(a_1) \dots u(i)$$

We observe that α contains a pattern of type 3142 corresponding to the indexes $r+1, s+1, s+2, d = \alpha^{-1}(u(a_1))$ and $\beta = \alpha(i,j)(r+1, s+1, d, s+2)$, as we wanted proof.

If u contains a pattern of type 3421 and $v = u(i,j)(a_1, a_4, a_3, a_2)$ then we have the following pair:

$$u = \dots u(i) \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(a_4) \dots u(j) \dots$$
$$v = \dots u(j) \dots u(a_4) \dots u(a_1) \dots u(a_2) \dots u(a_3) \dots u(i) \dots$$

We multiply by w_0:

$$vw_0 = u(i) \dots u(a_4) \dots u(a_2) \dots \boxed{u(a_1)} \dots \boxed{u(a_4)} \dots u(j) \dots$$
$$uw_0 = u(j) \dots u(a_3) \dots u(a_3) \dots \boxed{u(a_2)} \dots \boxed{u(a_1)} \dots u(i) \dots$$

We apply Lemma 2.10 to the common inversion between the boxed columns and we obtain that $(u,v) \sim (u_1, v_1)$ where:

$$u_1 = u(i) \dots u(a_4) \dots u(a_2) \dots u(a_4)\, u(a_1) \dots u(j)$$
$$v_1 = u(j) \dots u(a_3) \dots u(a_3) \dots u(a_1)\, u(a_2) \dots u(i)$$

And this is a pair of type 3. of Theorem 3.17. $\square$

Now we want to give an example of application of our formulas to a class of permutations which is not equivalent to the ones of Theorem 3.3, but it is related to these.

Consider $u = 134256$ and $v = 623541$ since $\tilde{R}_{u,v}(t) = t^6(1+t^2)(2+t^2)$ then (u,v) cannot be equivalent to a pair of permutations satisfying Theorem 3.3 or one of equivalent class. By Theorem 1.17 applied to the descent $(4,5) \in v$, since $(4,5) \notin u$ we have:

$$\tilde{R}_{u,v}(t) = \tilde{R}_{134526,623451}(t) + t\tilde{R}_{134256,623451}(t)$$

Note the the first summand satisfies the conditions of Theorem 3.17, while the second summand satisfies the conditions of Theorem 3.12. Then

$$\tilde{R}_{u,v}(t) = t^6(1+t^2) + tt^5(1+t^2)$$

We have seen an example of the irreducible class described in the following proposition:

Proposition 3.21. *Let $u, v \in S_n$, $1 \le i < a_1 < a_2 < a_3 < a_4 < j \le n$, be such that: u contains a pattern 2314 at the positions a_1, a_2, a_3, a_4 and $v = u(i,j)(a_1, a_3, a_4, a_2)$; $\forall s \in [a_3, a_4]$, $u(s) \notin [u(a_3), u(a_4)]$, then*

$$\tilde{R}_{u,v}(t) = t^6(1+t^2)^{\frac{inv(v)-inv(u)-8}{2}}(2+t^2)$$

Proof. Let u be such that contains a pattern 2314 at the positions a_1, a_2, a_3, a_4 and $v = u(i,j)(a_1, a_3, a_4, a_2)$ $\forall s \in [a_3, a_4]$, $u(s) \notin [u(a_3), u(a_4)]$.

$$\begin{array}{l} u = u(i) \dots u(a_1) \dots u(a_2) \dots \boxed{\begin{array}{c} u(a_3) \\ u(a_4) \end{array}} u(a_3+1) \dots u(t)\, u(t+1) \dots u(a_4-1)\, u(a_4) \dots u(j) \\ v = u(j) \dots u(a_3) \dots u(a_1) \dots \phantom{\boxed{u(a_4)}} u(a_3+1) \dots u(t)\, u(t+1) \dots u(a_4-1)\, u(a_2) \dots u(i) \end{array}$$

By Theorem 1.17 we can suppose that the values between $u(a_3)$ and $u(a_4)$. Define $t = max\{s \in [a_3+1, a_4-1] : u(s) < u(a_3)\}$, then $u(a_3+1) < \dots < u(t) < u(a_3) < u(a_4) < u(t+1) < \dots u(a_4-1)$. Then $(u,v) \sim (u_1, v_1)$, where $u_1 = us_{a_3} \dots s_{t-1}s_{t+1} \dots s_{a_4-1}$ and $v_1 = vs_{a_3} \dots s_{t-1}s_{t+1} \dots s_{a_4-1}$:

$$\begin{array}{l} u_1 = u(i) \dots u(a_1) \dots u(a_2) \dots u(a_3+1) \dots u(t) \boxed{\begin{array}{c} u(a_3) \\ u(a_4) \end{array}} u(a_4)\, u(t+1) \dots u(a_4-1) \dots u(j) \\ v_1 = u(j) \dots u(a_3) \dots u(a_1) \dots u(a_3+1) \dots u(t) \phantom{\boxed{u(a_4)}} u(a_2)\, u(t+1) \dots u(a_4-1) \dots u(i) \end{array}$$

By Theorem 1.17 applied to the descent $s \in (v_1^{-1}(u(a_3)), v_1^{-1}(u(a_2)))$, since $s \in D(v_1)$, but $s \notin D(u_1)$ we have:

$$\tilde{R}_{u_1,v_1}(t) = \tilde{R}_{u_1 s, v_1 s}(t) + t\tilde{R}_{u_1, v_1 s}(t)$$

Note that:

$$\begin{array}{l} u_1 s = u(i) \dots u(a_1) \dots u(a_2) \dots u(a_3+1) \dots u(t)\, u(a_4)\, u(a_3)\, u(t+1) \dots u(a_4-1) \dots u(j) \\ v_1 s = u(j) \dots u(a_3) \dots u(a_1) \dots u(a_3+1) \dots u(t)\, u(a_2)\, u(a_4)\, u(t+1) \dots u(a_4-1) \dots u(i) \end{array}$$

and this is a pair of type 2. of Theorem 3.17, moreover $inv(u_1) = inv(u_1 s) + 1$ and $inv(v_1 s) = inv(v_1) - 1$.

$$\begin{array}{l} u_1 = u(i) \dots u(a_1) \dots u(a_2) \dots u(a_3+1) \dots u(t)\, u(a_3)\, u(a_4)\, u(t+1) \dots u(a_4-1) \dots u(j) \\ v_1 s = u(j) \dots u(a_3) \dots u(a_1) \dots u(a_3+1) \dots u(t)\, u(a_2)\, u(a_4)\, u(t+1) \dots u(a_4-1) \dots u(i) \end{array}$$

and this is a pair of type 2 of Theorem 3.12.

Thus $\tilde{R}_{u,v}(t) = t^6(1+t^2)^{\frac{inv(v)-inv(u)-8}{2}} + t^6(1+t^2)^{\frac{inv(v)-inv(u)-6}{2}}$, and this concludes the proof of this part.

□

By Theorem 1.16 we have that:

Corollary 3.22. *Let $u, v \in S_n$, $1 \le i < a_1 < a_2 < a_3 < a_4 < j \le n$, be such that: u contains a pattern 2314 at the positions a_1, a_2, a_3, a_4 and $v = u(i,j)(a_1, a_3, a_4, a_2)$; $\forall s \in [a_3, a_4]$, $u(s) \notin [u(a_3), u(a_4)]$, then*

$$R_{u,v}(t) = (q-1)^6(1-q+q^2)^{\frac{inv(v)-inv(u)-8}{2}}(1+q^2)$$

3.5 How many explicit formulas do we have actually found?

In Chapter 2 we have seen that from the irreducible classes related to $(2,1,1,\dots)$ and $(3,1,1,\dots)$, we obtain formulas for infinitely many irreducible classes, for example $(3,2,1,\dots)$, but also for $(3,\dots,3,1,1,\dots)$, $(2,\dots,2,1,1\dots)$, or $(3,\dots,3,2\dots,2,1,\dots)$. Here we observe that by using the results collected in this chapter we have a large amount of partitions for which we can classify related reducible classes using Theorem 1.21. We said that this theorem can be viewed, together with all the irreducible classes that we have classified in this thesis, as an explicit formula generating machinery; here we write explicitly, as an example, one closed product formula for the partition (k,m,2,2,1,1...).

Proposition 3.23. *Let $u \in S_n$ and $1 \leq i < a_1 < a_2 < a_3 < \cdots < a_k < j < c < b_1 < b_2 < \cdots < b_m < d \leq n$, such that u contains a pattern $23\dots(k-1)k1$ at the positions a_1, a_2, ..., a_k and a pattern $m12\dots m-1$ at the positions $b_1, b_2, \dots, b_m$. If $u < v = u(i,j)(c,d)(a_1, a_k, a_{k-1}, \dots, a_2)(b_1, b_2, \dots, b_m)$ then*

$$\tilde{R}_{u,v}(t) = t^{m+k+4}(1+t^2)^{\frac{inv(v)-inv(u)-(m+k+4)}{2}}$$

Proof. It follows from Theorem 1.21, Theorem 3.3 and Theorem 3.5. □

Moreover we have seen in Proposition 3.14 and 3.21 that for certain irreducible classes it is possible to obtain explicit formulas, not by equivalence, as in the main results, but by proving that the $\tilde{R}$-polynomial associated to this is a linear combinations on $\mathbb{Z}[t]$ of $\tilde{R}$-polynomials of classified classes. We have many experimental evidences of the fact that there is a large number of R-polynomials of this kind and all these considerations enforce the idea that explicit formulas give other explicit formulas. It is clear that this can be generalized to every pair and that the knowledge of a large number of explicit formulas makes the computation of all the R-polynomials easier.

Chapter 4

Open problems and conjectures

4.1 The R-polynomials of cycle type $(4, 1, 1, 1, \dots)$ and others in $(k, 2, 1, 1 \dots)$

We propose a natural prosecution of the investigation on the irreducible classes and we can classify this section as work in progress. Following section 2.2, a good question is if Theorem 1.20 permits to completely solve the problem of irreducible class of cycle type $(4, 1, 1, 1, \dots)$, i.e. the case in which uv^{-1} is a 4-cycle.

The problem of two permutations "which differ" for k-cycles nested in a transposition is treated in Chapter 3 where we followed the idea that we can summarize in the "pattern rule" or "law of natural reordering of a pattern". As we said before, the law of natural reordering of a pattern consists in the generalization of what happen for the classes studied in Chapter 3: if a permutation u contains a pattern of type $k12 \dots k-1$ or $23 \dots k1$ between two fixed positions and v is obtained from u by applying the cycle which put the elements of the pattern in their natural order in u, then the R-polynomial associated to (u, v) factors nicely. This law has sense for patterns which are not increasing subsequences in a permutation and it seems that there is a "quite mysterious correspondence" with this law and the existence of explicit formulas, which has been showed in this thesis. As the cases studied for $k = 3$ and for $k = 4$ suggest by symmetry it should be possible to classify other classes related to pattern of length k, following the pattern law and not only (see Proposition 3.17 and Corollary 3.12).

4.2 Other Coxeter groups and the equivalence relation

We briefly explain other possible problems around the ones studied in this work. The reduction process on which are based the proof of our main results can be expressed in theoretical algebraic terms of an equivalence relation which is valid in every finite Coxeter group. Therefore is natural thinking to a possible extension of our formulas to other finite Coxeter groups. So far we do not know how can be reformulated our statements and the law of natural reordering in a general context.
We think that first step to approach this matter is the study of the Weyl groups of type B_n, for which is also known a characterization of the Conjugacy classes in terms of double partitions.

4.3 Explicit calculation of the Kazhdan-Lusztig polynomials

In this last section we present another prosecution of this research which is related to one of the main motivations for which the R-polynomials are studied. It is the application of the results on R-polynomials to the explicit calculation of Kazhdan-Lusztig polynomials.

To explain this concept we give next theorem definition:

Theorem 4.1. *Let $u, v \in W$.*
There exists a unique family of polynomials
$\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$*, such that*

(i) $P_{u,v}(q) = 0$*, if* $u \not\leq v$

(ii) $P_{u,v}(q) = 1$*, if* $u = v$

(iii) $deg(P_{u,v}(q) \leq \frac{1}{2}(\ell(v) - \ell(u) - 1)$ *if* $u < v$

(iv)
$$q^{\ell(v)-\ell(u)} P_{u,v}(\frac{1}{q}) = \sum_{a \in [u,v]} R_{u,a}(q) P_{a,v}(q)$$

if $u \leq v$

A proof of this theorem can be find in [10] Chapter 9.

The family $\{P_{u,v}(q)\}_{u,v\in W} \subseteq \mathbb{Z}[q]$ whose existence and uniqueness are guaranteed by the previous theorem are called the *Kazhdan-Lusztig polynomials.* These polynomials play an important role in the study of the geometric properties of Schubert varieties as well as the representation theory of semisimple Lie algebras, see, for example,[12], [5], [1]. Explicit formulas are known for these polynomials see [13], [14],[6].

We conclude with a significant facts on the geometrical meaning of Kazhdan-Lusztig.

For $w \in S_n$ let Ω_w be the *Schubert cell* indexed by w, and we define the Schubert variety indexed by v to be $\bar{\Omega_v}$ (Zariski closure).

It is a well known property of Bruhat order the following:

$$\bar{\Omega_u} \subseteq \bar{\Omega_v} \Longleftrightarrow u \leq v$$

See e.g. [9]

Theorem 4.2 (Kazhdan-Lusztig, 1989). *Let $u, v \in S_{n+1}$, $u \leq v$. Then*

$$P_{u,v}(q) = \sum_{i\geq 0} q^i dim_{\mathbb{C}}(IH^{2i}(\bar{\Omega_v}, \mathbb{C})_{\Omega_u})$$

where $IH^{2i}(\bar{\Omega_v}, \mathbb{C})_{\Omega_u}$ is the local intersection cohomology of $\bar{\Omega_v}$, at a point of Ω_u

This is [12], Theorem 4.3.

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